The product BMO space and applications

Emil Airta

Master's thesis

UNIVERSITY OF HELSINKI

Department of Mathematics and Statistics
University of Helsinki

May 24, 2018

HELSINGIN YLIOPISTO — HELSINGFORS UNIVERSITET — UNIVERSITY OF HELSINKI

Tiedekunta/Osasto — Fakultet/Sektion — Faculty	Laitos — Institution — Department
Matemaattis-luonnontieteellinen	Matematiikan ja tilastotieteen laitos

Tekijä — Författare — Author

Emil Airta

Työn nimi — Arbetets titel — Title

The product BMO space and applications

Oppiaine — Läroämne — Subject

Matematiikka

Työn laji — Arbetets art — Level	Aika — Datum — Month and year	Sivumäärä — Sidoantal — Number of pages
Pro gradu -tutkielma	Toukokuu 2018	51 s.

Tiivistelmä — Referat — Abstract

Tutkielman aiheena on tutkia kahden parametrin analyysiä, erityisesti tuloavaruuden BMO:ta, Calderón-Zygmund operaattoreita ja kahden parameterin paratuloja.

Kappaleessa kaksi esitellään merkinnät, perusmääritelmiä ja perustuloksia. Tärkeitä työkaluja kuten dyadiset systeemit, tuplamaksimaalifunktiot sekä Haarin esitystä L^p-funktioille tarvitaan läpi tutkielman. Kun tutkitaan singulaarisia integraaleja tulee tarpeelliseksi määritellä BMO-avaruudet. Yhden parametrin tapauksessa osoittauttuu, että määritelmä voidaan tehdä muutamalla yhtäpitävällä tavalla. Tilanne on toinen käsiteltäessä kahden parametrin BMO-avaruutta. Luontevimmalla yhden parametrin BMO-avaruuden määritelmällä ei saada helposti kahden parametrin versiota. Lisäksi analogisin vaihtoehto ei edes tuota haluttua BMO-avaruutta.

Kappale kolme käsittelee Calderón-Zygmund operaattoreita. Aluksi määritellään yhden parametrin Calderón-Zygmund operaattorit, jonka jälkeen vasta voidaan määritellä kahden parametrin Calderón-Zygmund operaattorit. Työn yksi päätulos on näyttää $T1 \in BMO_{prod}$, missä T on L^2 rajoitettu kahden parametrin Calderón-Zygmund operaattori. Tämän osiottamiseksi, joudutaan esittämään tuloavaruuden eräs peitelause, nimeltään Journén peitelause.

Tutkielman viimeisessä kappaleessa tutkitaan kahden parametrin paratulojen rajoittuneisuutta. Nämä paratulot ovat oleellinen osa kahden parametrin $T1$ lauseiden todistusta, joita tässä työssä ei käsitellä niiden vaativuuden takia.

Avainsanat — Nyckelord — Keywords

Harmoninen analyysi, BMO -avaruus, Calderón-Zygmund operaattorit

Säilytyspaikka — Förvaringsställe — Where deposited

Kumpulan tiedekirjasto

Muita tietoja — Övriga uppgifter — Additional information

Contents

Chapter 1

Introduction

In this thesis, we study bi-parameter analysis, namely properties of the product BMO spaces, the bi-parameter Calderon-Zygmund operators and boundedness of the bi-parameter paraproducts. These topics are related to an active field in harmonic analysis.

In the second chapter, we introduce notation, basic definitions and basic results such as dyadic grids, maximal functions, Haar functions and martingale difference representation. As in many topics in harmonic analysis, the dyadic systems or grids are an essential tool. We also introduce tools, such as the double maximal function, related to bi-parameter analysis. The notation is obtained from preprint by Li, Martikainen and Vuorinen [4].

In addition, when dealing with singular integrals, we need to define BMO spaces in spaces $\mathbb{R}^n$ and $\mathbb{R}^{n+m}$. In the one-parameter settings we can define the BMO space in a few different ways and in fact the standard definition is not the most useful for us. In the product space the definition is not completely obvious. Results later on in this thesis show that the correct space is not the most natural analogue of the one-parameter BMO space. As a basic reference we use [4] and the book by Muscalu and Schlag [6].

Then in the third chapter we discuss some of the basic properties of the product BMO spaces. In the first section we show that the John-Nirenberg result holds also in the bi-parameter setting, that is, the general product BMO norms are equivalent for every exponent $0 < p < q < \infty$. Since we have two different BMO definitions, the product BMO and the rectangular BMO, we want to show that these two spaces are not equal. For this we show the well-known Carleson's counterexample. These topics are discussed in the book [6]. The last subject in the chapter is an estimate related to BMO sequences. We take the proof of this well-known $H^1 - \text{BMO}$ type duality estimate from [5].

For applications of the product BMO space, in the fourth chapter we study Calderon-Zygmund operators. The main result of this chapter is $T1 \in \text{BMO}_{prod}(\mathbb{R}^n \times \mathbb{R}^m)$, where T is a bi-parameter Calderón-Zygmund operator such that $T : L^2 \to L^2$. The dyadic $T1 \in \text{BMO}_{prod}$ proof is adapted from [1],[4]. See also the original reference [2]. Naturally, even the definition of the bi-parameter Calderón-Zygmund operators needs the one-parameter Calderón-Zygmund operator. For the one-parameter Calderón-Zygmund operator we are going to show $L^\infty \to \mathrm{BMO}$ boundedness, in particular, we have $U1 \in \mathrm{BMO}(\mathbb{R}^n)$, where U is a Calderón-Zygmund operator such that $U : L^2 \to L^2$. The proof of this theorem is a straightforward calculation but, as we see in the last section of chapter 4, the situation is completely different in the bi-parameter setting. In the bi-parameter case we need a covering lemma called Journé's lemma [3]. This can be proved with an arbitrary product measure as shown by Hytönen and Martikainen [1]. However, we simply write the proof with Lebesque measure.

In the last chapter, we introduce bi-parameter paraproducts. These paraproducts appear in bi-parameter $T1$ theorems and it is essential that these paraproducts are bounded. These $T1$ theorems are too advanced for this thesis but we are going to study the necessary condition for the boundedness of the paraproducts. The boundedness properties are also discussed in [5].

Acknowledgements. I would like to thank Henri Martikainen for excellent explanations of the topics and for supervision of this thesis.

Chapter 2

Basic definitions and results

2.1 Vinogradov notation

We denote $A \lesssim B$ if $A \leq CB$ for some constant depending only on the dimension of the underlying space, on integration exponents and on other absolute constants appearing in the assumptions. Then naturally $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

2.2 Dyadic notation

We say that $\mathcal{D}^n$ is a dyadic grid on $\mathbb{R}^n$ if $\mathcal{D}^n = \bigcup_{k\in\mathbb{Z}} \mathcal{D}^n_k$ such that for every $k \in \mathbb{Z}$, $\mathcal{D}^n_k$ is a collection of half-open pairwise disjoint cubes with following properties:

- For every cube $I \in \mathcal{D}^n_k$, side-length of I is 2^{-k}
- $\mathbb{R}^n = \bigcup_{I\in\mathcal{D}^n_k} I$
- $$I = \bigcup_{\substack{I'\in\mathcal{D}^n_{k+1} \\ I'\subset I}} I'$$
  for every cube $I \in \mathcal{D}^n_k$.

Example of such a dyadic grid is

$$\left\{2^{-k}([0,1)^n + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\right\}.$$

We say that a cube $I \subset \mathbb{R}^n$ is dyadic cube if $I \in \mathcal{D}^n$ for some dyadic grid $\mathcal{D}^n$. For a cube $I \subset \mathbb{R}^n$, $\ell(I)$ denotes the side-length of I and if I is a dyadic cube, then $\mathrm{ch}(I)$ denotes the dyadic children of I, that is, if $\ell(I) = 2^{-k}$, then for all $I' \in \mathrm{ch}(I)$ holds that $I' \subset I$ and $\ell(I') = 2^{-k-1}$. In addition, we denote the unique dyadic cube that contains a point x and has side-length of 2^{-k} by $I_k(x)$.

Throughout of this thesis instead of the standard Euclidean distance, $|x-y|$ denotes $\|x-y\|_{\ell^\infty}$. Since these norms are equivalent, it does not matter which one we use. For convenience calculations are more clear if cubes have equal diameter and side-length.

2.3 Bi-parameter notation

When working in the product space, every $x \in \mathbb{R}^{n+m}$ (or y) is always a tuple (x_1, x_2), where $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$. Moreover, we denote cubes in $\mathbb{R}^n$ by I and cubes in $\mathbb{R}^m$ by J. For dyadic grids we denote grids in $\mathbb{R}^n$ by $\mathcal{D}^n$ and in $\mathbb{R}^m$ by $\mathcal{D}^m$.

Working with bi-parameter functions we often need integral pairing with respect to only one of the two variables. Thus we define $\langle f, g\rangle_1 : \mathbb{R}^m \to \mathbb{C}$ for measurable functions $f : \mathbb{R}^{n+m} \to \mathbb{C}$ and $g : \mathbb{R}^n \to \mathbb{C}$ by

$$\langle f, g\rangle_1(x_2) := \langle f(\cdot, x_2), g\rangle = \int_{\mathbb{R}^n} f(x_1, x_2)\overline{g(x_1)}\,\mathrm{d}x_1$$

and similarly for $h : \mathbb{R}^m \to \mathbb{C}$, we define $\langle f, h\rangle_2 : \mathbb{R}^n \to \mathbb{C}$ by

$$\langle f, h\rangle_2(x_1) := \langle f(x_1, \cdot), h\rangle.$$

If we have integral over both variables, usually it is convenient to use only one integral sign and only if needed we use the double integral sign.

Furthermore, we need the support of a bi-parameter function only in one of the underlying spaces. Thus for a bi-parameter function f we denote

$$\mathrm{spt}_{\mathbb{R}^n} f := \overline{\{x_1 \in \mathbb{R}^n : \exists x_2 \in \mathbb{R}^m, f(x_1, x_2) \neq 0\}}.$$

2.4 Maximal functions

We say that $M_{\mathbb{R}^n}$ is the maximal function in $\mathbb{R}^n$ defined by

$$M_{\mathbb{R}^n} f(x_1) = \sup_{I : x_1 \in I} \frac{1}{|I|} \int_I |f(y_1)|\,\mathrm{d}y_1,$$

where the supremum is taken over all possible cubes containing point x_1.

Let $\mathcal{D}^n$ be a dyadic grid in $\mathbb{R}^n$. We say that $M_{\mathcal{D}^n}$ is the dyadic maximal function defined by

$$M_{\mathcal{D}^n} f(x_1) = \sup_{I : x_1 \in I} \frac{1}{|I|} \int_I |f(y_1)| \, \mathrm{d}y_1,$$

where the supremum is taken over all possible dyadic cubes cubes in $\mathcal{D}^n$ containing point x_1.

Recall the basic result for the one-parameter maximal function.

Lemma 2.1. *It holds*

$$\|M_{\mathbb{R}^n} f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

for $p \in (1, \infty)$.

For a locally integrable function f on $\mathbb{R}^{n+m}$ we define the double maximal function by

$$Mf(x_1, x_2) := \sup_{R : (x_1, x_2) \in R} \frac{1}{|R|} \iint_R |f(y_1, y_2)| \, \mathrm{d}y_1 \, \mathrm{d}y_2, \tag{2.2}$$

where the supremum is taken over all possible rectangles containing point $x = (x_1, x_2)$.

Let $\mathcal{D} := \mathcal{D}^n \times \mathcal{D}^m$ be a grid of dyadic rectangles in the product space $\mathbb{R}^n \times \mathbb{R}^m$. Then we define the dyadic double maximal operator by

$$M_{\mathcal{D}} f(x_1, x_2) := \sup_{R : (x_1, x_2) \in R} \frac{1}{|R|} \iint_R |f(y_1, y_2)| \, \mathrm{d}y_1 \, \mathrm{d}y_2, \tag{2.3}$$

where the supremum is taken over all possible dyadic rectangles in $\mathcal{D}$ containing point $x = (x_1, x_2)$.

Notice that by definition we have

$$M_{\mathcal{D}} f(x_1, x_2) \lesssim Mf(x_1, x_2).$$

Lemma 2.4. *There holds*

$$\|Mf\|_{L^p(\mathbb{R}^{n+m})} \lesssim \|f\|_{L^p(\mathbb{R}^{n+m})} \tag{2.5}$$

for all $p \in (1, \infty)$.

Proof. We need to show that the double maximal operator is bounded in L^p. First, notice that

$$\frac{1_I(x_1)}{|I|} \frac{1_J(x_2)}{|J|} \int_I \int_J f(y_1, y_2) \, \mathrm{d}y_2 \, \mathrm{d}y_1 \leq \frac{1_I(x_1)}{|I|} \int_I M_{\mathbb{R}^m}(f(y_1, \cdot))(x_2) \, \mathrm{d}y_1.$$

Thus we have that

$$\begin{aligned}
Mf(x_1, x_2) &= \sup_{R=I\times J} \frac{1_I(x_1)}{|I|}\frac{1_J(x_2)}{|J|}\int_I\int_J f(y_1,y_2)\,\mathrm{d}y_2\,\mathrm{d}y_1\\
&\le M_{\mathbb{R}^n}(M_{\mathbb{R}^m} f(\cdot, x_2))(x_1)\\
&= M^1_{\mathbb{R}^n}\circ M^2_{\mathbb{R}^m} f(x_1,x_2),
\end{aligned}$$

where we denote $M^1_{\mathbb{R}^n}$ and $M^2_{\mathbb{R}^m}$ by

$$M^1_{\mathbb{R}^n} f(x_1,x_2) := M_{\mathbb{R}^n}(f(\cdot,x_2))(x_1)$$

and

$$M^2_{\mathbb{R}^m} f(x_1,x_2) := M_{\mathbb{R}^m}(f(x_1,\cdot))(x_2).$$

Since we know that the classical maximal operators have the desired property, we can check the boundedness of $M^1_{\mathbb{R}^n}\circ M^2_{\mathbb{R}^m}$. Thus we have by Fubini's theorem

$$\begin{aligned}
\int_{\mathbb{R}^{n+m}} (Mf(x,y))^p\,\mathrm{d}x\,\mathrm{d}y &\le \int_{\mathbb{R}^m}\int_{\mathbb{R}^n}\left(\left(M^1_{\mathbb{R}^n}\circ M^2_{\mathbb{R}^m}\right)f(x,y)\right)^p\,\mathrm{d}x\,\mathrm{d}y\\
&\le C\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}\left(M^2_{\mathbb{R}^m} f(x,y)\right)^p\,\mathrm{d}x\,\mathrm{d}y\\
&= C\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}\left(M^2_{\mathbb{R}^m} f(x,y)\right)^p\,\mathrm{d}y\,\mathrm{d}x\\
&\le C'\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}|f(x,y)|^p\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

from which we conclude that

$$\|Mf\|_{L^p(\mathbb{R}^{n+m})} \lesssim \|f\|_{L^p(\mathbb{R}^{n+m})}.$$

□

2.5 Haar functions

First, we consider the dimension one. For every interval I there are two Haar functions associated to it: the non-cancellative $h^0_I := |I|^{-1/2}1_I$ and the cancellative $h^1_I := |I|^{-1/2}(1_{I_l} - 1_{I_r})$ where I_l is the left half and I_r is the right half of the interval I.

Now we can consider the general case. For a cube $I = I_1\times I_2\times\cdots\times I_n\subset\mathbb{R}^n$, we have the Haar functions as product of the one-dimensional Haar functions

$$h^\eta_I(x) = h^{\eta_1,\ldots,\eta_n}_{I_1\times\cdots\times I_n}(x_1,\ldots,x_n) := \prod_{i=1}^n h^{\eta_i}_{I_i}(x_i).$$

Here we have the non-cancellative term $h_I^0(x) = |I|^{-1/2} 1_I(x)$ similarly as in the dimension one and the cancellative terms satisfies

$$\int h_I^\eta = 0,$$

where $\eta \in \{0,1\}^n \setminus \{0\}$.

In addition, let $I', I \in \mathcal{D}^n$ and $\eta, \epsilon \in \{0,1\}^n$ for fixed $\mathcal{D}^n$. Then

$$\langle h_{I'}^\epsilon, h_I^\eta \rangle = \delta_{I,I'} \delta_{\eta,\epsilon},$$

where

$$\delta_{a,b} = \begin{cases} 1 & \text{for } a = b, \\ 0 & \text{for } a \neq b. \end{cases}$$

Hence, $\{h_I^\eta\}_{I \in \mathcal{D}^n, \eta \in \{0,1\}^n}$ is an orthonormal sequence in L^2.

2.6 Martingale difference representation

We denote the average of a locally integrable function f in $I \subset \mathbb{R}^n$ by

$$\langle f \rangle_I := \frac{1}{|I|} \int_I f(x_1) \, \mathrm{d}x_1.$$

For $I \in \mathcal{D}^n$ and a locally integrable function $f : \mathbb{R}^n \to \mathbb{C}$, we have

$$\sum_{\eta \in \{0,1\}^n \setminus \{0\}} \langle f, h_I^\eta \rangle h_I^\eta = \Delta_I f$$

where Δ_I is the martingale difference defined as

$$\Delta_I f = \sum_{I' \in \mathrm{ch}(I)} \big(\langle f \rangle_{I'} - \langle f \rangle_I \big) 1_{I'}.$$

Remark 2.6. For every $I \in \mathcal{D}^n$ the martingale difference $\Delta_I f$ has mean zero, namely

$$\begin{aligned} \int \Delta_I f &= \sum_{I' \in \mathrm{ch}(I)} \int \big(\langle f \rangle_{I'} - \langle f \rangle_I \big) 1_{I'} \\ &= \sum_{I' \in \mathrm{ch}(I)} \int \langle f \rangle_{I'} 1_{I'} - \sum_{I' \in \mathrm{ch}(I)} \int \langle f \rangle_I 1_{I'} \\ &= \int_I f - \int_I f = 0. \end{aligned}$$

Remark 2.7. $\Delta_I f$ is constant in every dyadic child of I. Moreover, by this fact and previous remark $\Delta_I \Delta_{I'} f = 0$ whenever $I \neq I'$ and $\Delta_I \Delta_I f = \Delta_I f$.

Remark 2.8. Suppose that $I, I' \in \mathcal{D}^n$ such that $I \neq I'$. Clearly, if $I \cap I' = \emptyset$, then $\langle \Delta_I f, \Delta_{I'} f \rangle = 0$. Thus the pairing can be non-vanishing only if $I \subsetneq I'$ or $I' \subsetneq I$. Hence, we may assume $I \subsetneq I'$. Then by previous remarks, we get

$$\langle \Delta_I f, \Delta_{I'} f \rangle = \left(\overline{\langle f \rangle}_{I_0} - \overline{\langle f \rangle}_{I'} \right) \int \Delta_I f = 0,$$

where $I_0 \in \mathrm{ch}(I')$ such that $I_0 \subset I$. Thus we conclude that $\langle \Delta_I f, \Delta_{I'} f \rangle = 0$ if $I \neq I'$.

Lemma 2.9. *Let $1 \leq p < \infty$. There holds*

$$f = \sum_{I \in \mathcal{D}^n} \Delta_I f$$

almost everywhere for $f \in L^p(\mathbb{R}^n)$.

Proof. Let $\mathcal{D}^n$ be a dyadic grid in $\mathbb{R}^n$. Fix $x_1 \in \mathbb{R}^n$ and let $k, K \in \mathbb{Z}$ such that $k > K$. Since for every $l \in \mathbb{Z}$ there exists a unique dyadic cube $I_l(x_1) \in \mathcal{D}^n_l$ containing point x_1, we have

$$\begin{aligned}
\sum_{\substack{I \in \mathcal{D}^n \\ 2^{-k} < \ell(I) \leq 2^{-K}}} \Delta_I f(x_1) &= \sum_{\substack{I \in \mathcal{D}^n \\ 2^{-k} < \ell(I) \leq 2^{-K}}} \sum_{I' \in \mathrm{ch}(I)} \left(\langle f \rangle_{I'} - \langle f \rangle_I \right) 1_{I'}(x_1) \\
&= \langle f \rangle_{I_k(x_1)} \\
&\quad - \langle f \rangle_{I_{k-1}(x_1)} + \langle f \rangle_{I_{k-1}(x_1)} \\
&\quad \vdots \\
&\quad - \langle f \rangle_{I_{K+1}(x_1)} + \langle f \rangle_{I_{K+1}(x_1)} \\
&\quad - \langle f \rangle_{I_K(x_1)} \\
&= \langle f \rangle_{I_k(x_1)} - \langle f \rangle_{I_K(x_1)}.
\end{aligned}$$

Hence, we estimate

$$|\langle f \rangle_{I_K(x_1)}| \leq (2^{\frac{n}{p}})^K \, \|f\|_{L^p}$$

and we get $\langle f \rangle_{I_K(x_1)} \to 0$ as $K \to -\infty$. On the other hand,

$$\lim_{k \to \infty} \langle f \rangle_{I_k(x_1)} = f(x_1)$$

for almost every $x_1 \in \mathbb{R}^n$ by Lebesgue differentiation theorem. Thus we have

$$\sum_{I \in \mathcal{D}^n} \Delta_I f(x_1) = f(x_1)$$

for almost every $x_1 \in \mathbb{R}^n$. $\square$

In addition, we need that the martingale difference representation of a function f converges to f in L^p.

Lemma 2.10. *Let $\mathcal{D}^n$ be a dyadic grid. For $p \in (1, \infty)$ and $f \in L^p(\mathbb{R}^n)$ there holds*

$$\lim_{k\to\infty} \Big\| \sum_{\substack{I\in\mathcal{D}^n \\ 2^{-k}<\ell(I)\le 2^k}} \Delta_I f - f \Big\|_{L^p(\mathbb{R}^n)} = 0.$$

Proof. Observe that

$$\Big| \sum_{\substack{I\in\mathcal{D}^n \\ 2^{-k}<\ell(I)\le 2^k}} \Delta_I f(x_1) \Big| = |\langle f\rangle_{I_k(x_1)} - \langle f\rangle_{I_{-k}(x_1)}| \le 2M_{\mathbb{R}^n} f(x_1)$$

for each fixed $k \in \mathbb{N}$ and for all $x_1 \in \mathbb{R}^n$. Thus we have maximal function as a dominant for the sequence and $\|M_{\mathbb{R}^n} f\|_{L^p} \lesssim \|f\|_{L^p}$. Hence, by Lemma 2.9 and dominated convergence theorem

$$\lim_{k\to\infty} \Big\| \sum_{\substack{I\in\mathcal{D}^n \\ 2^{-k}<\ell(I)\le 2^k}} \Delta_I f - f \Big\|_{L^p(\mathbb{R}^n)} = \Big\| \sum_{I\in\mathcal{D}^n} \Delta_I f - f \Big\|_{L^p(\mathbb{R}^n)} = 0.$$

$\square$

Lemma 2.11. *Let $\mathcal{D}^n$ be a dyadic grid. There holds*

$$\Big\| \Big(\sum_{I\in\mathcal{D}^n} |\Delta_I f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$$

for all $f \in L^2(\mathbb{R}^n)$.

Proof. By Lemma 2.10 and Remark 2.8 we have

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \Big| \sum_{I\in\mathcal{D}^n} \Delta_I f \Big|^2 = \sum_{I\in\mathcal{D}^n} \int_{\mathbb{R}^n} |\Delta_I f|^2 = \Big\| \Big(\sum_{I\in\mathcal{D}^n} |\Delta_I f|^2 \Big)^{\frac{1}{2}} \Big\|^2_{L^2(\mathbb{R}^n)}.$$

$\square$

Previous theorem can be formulated for $f \in L^p(\mathbb{R}^n)$

$$\Big\| \Big(\sum_{I\in\mathcal{D}^n} |\Delta_I f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)} \tag{2.12}$$

for $p \in (1, \infty)$. To justify two different definitions of the BMO space, we need $p \in (1, \infty)$ but the main theorems in this thesis require only $p = 2$. Since this is purely for motivational purpose, we do not prove this. Usually with $p = 2$ we write

$$\|f\|_{L^2(\mathbb{R}^n)}^2 = \sum_{I \in \mathcal{D}^n} \|\Delta_I f\|_{L^2(\mathbb{R}^n)}^2.$$

Corollary 2.13. *Let $f \in L^2(\mathbb{R}^n)$ and $\mathcal{D}^n$ a dyadic grid in $\mathbb{R}^n$. Then*

$$\sum_{I \in \mathcal{D}^n} |\langle f, h_I \rangle|^2 = \|f\|_{L^2(\mathbb{R}^n)}^2$$

where the summation over $\eta \in \{0,1\} \setminus \{0\}$ is suppressed.

Proof. Recall that

$$\Delta_I f = \sum_{\eta \in \{0,1\} \setminus \{0\}} \langle f, h_I^\eta \rangle h_I^\eta$$

then by Theorem 2.11 we get

$$\|f\|_{L^2(\mathbb{R}^n)}^2 = \sum_{I \in \mathcal{D}^n} \int_{\mathbb{R}^n} |\Delta_I f|^2 = \sum_{I \in \mathcal{D}^n} \int_{\mathbb{R}^n} \left| \sum_{\eta \in \{0,1\} \setminus \{0\}} \langle f, h_I^\eta \rangle h_I^\eta \right|^2.$$

Now by the orthogonality of the Haar functions

$$\begin{aligned} \sum_{I \in \mathcal{D}^n} \int_{\mathbb{R}^n} \left| \sum_{\eta \in \{0,1\} \setminus \{0\}} \langle f, h_I^\eta \rangle h_I^\eta \right|^2 &= \sum_{I \in \mathcal{D}^n} \sum_{\eta \in \{0,1\} \setminus \{0\}} |\langle f, h_I^\eta \rangle|^2 \\ &= \sum_{\eta \in \{0,1\} \setminus \{0\}} \sum_{I \in \mathcal{D}^n} |\langle f, h_I^\eta \rangle|^2. \end{aligned}$$

Hence, we have

$$\|f\|_{L^2(\mathbb{R}^n)}^2 = \sum_{\eta \in \{0,1\} \setminus \{0\}} \sum_{I \in \mathcal{D}^n} |\langle f, h_I^\eta \rangle|^2$$

as desired. From this point on the summation over $\eta \in \{0,1\} \setminus \{0\}$ is understood implicitly and we write

$$\langle f, h_I \rangle h_I = \sum_{\eta \in \{0,1\} \setminus \{0\}} \langle f, h_I^\eta \rangle h_I^\eta.$$

□

2.7 Representation in the product space

In this section, we want to show that we have similar representation in the product space $\mathbb{R}^{n+m}$ as in $\mathbb{R}^n$. First, we define the bi-parameter martingale difference. Let $\mathcal{D}^n$ and $\mathcal{D}^m$ be some dyadic grids in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. For $I \times J \in \mathcal{D}^n \times \mathcal{D}^m$ we define

$$\Delta_{I\times J} f = \Delta_I^1(\Delta_J^2 f) = \Delta_J^2(\Delta_I^1 f),$$

where

$$\Delta_I^1 f(x_1, x_2) := \Delta_I(f(\cdot, x_2))(x_1)$$

and

$$\Delta_J^2 f(x_1, x_2) := \Delta_J(f(x_1, \cdot))(x_2).$$

Notice that $\Delta_I^1 f = h_I \otimes \langle f, h_I\rangle_1$ and $\Delta_J^2 f = \langle f, h_J\rangle_2 \otimes h_J$. Hence, we have

$$\begin{aligned}
\Delta_{I\times J} f &= h_I \otimes \langle \Delta_J^2 f, h_I\rangle_1 \\
&= \big(\langle\langle f, h_J\rangle_2, h_I\rangle_1\big) h_I \otimes h_J \\
&= \Big(\int_{\mathbb{R}^n} \langle f, h_J\rangle_2 h_I\Big) h_I \otimes h_J \\
&= \Big(\int_{\mathbb{R}^{n+m}} f[h_I \otimes h_J]\Big) h_I \otimes h_J = \langle f, h_R\rangle h_R
\end{aligned}$$

where $R = I \times J$, $h_R := h_I \otimes h_J$.

Lemma 2.14. *Let $p \in (1, \infty)$. There holds*

$$f = \lim_{\substack{l\to\infty \\ k\to\infty}} \sum_{\substack{I\times J\in\mathcal{D}^n\times\mathcal{D}^m \\ 2^{-l}<\ell(I)\le 2^l \\ 2^{-k}<\ell(J)\le 2^k}} \Delta_{I\times J} f$$

almost everywhere for $f \in L^p(\mathbb{R}^n)$.

Proof. Fix $(x_1, x_2) \in \mathbb{R}^{n+m}$. Similarly as in the one-parameter case we write

$$\begin{aligned}
&\sum_{\substack{J\in\mathcal{D}^m \\ 2^{-k}<\ell(J)\le 2^k}} \sum_{\substack{I\in\mathcal{D}^n \\ 2^{-l}<\ell(I)\le 2^l}} \Delta_I^1(\Delta_J^2 f)(x_1, x_2) \\
&= \sum_{\substack{J\in\mathcal{D}^m \\ 2^{-k}<\ell(J)\le 2^k}} \Big(\langle \Delta_J^2 f(\cdot, x_2)\rangle^1_{I_l(x_1)} - \langle \Delta_J^2 f(\cdot, x_2)\rangle^1_{I_{-l}(x_1)}\Big) \qquad (2.15) \\
&= \langle f\rangle_{I_l(x_1)\times J_k(x_2)} - \langle f\rangle_{I_l(x_1)\times J_{-k}(x_2)} - \langle f\rangle_{I_{-l}(x_1)\times J_k(x_2)} + \langle f\rangle_{I_{-l}(x_1)\times J_{-k}(x_2)} \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

For term IV, as previously, we get

$$|\langle f\rangle_{I_{-l}(x_1)\times J_{-k}(x_2)}| \le |I_{-l}(x_1)\times J_{-k}(x_2)|^{-\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^{n+m})}.$$

Hence,

$$\lim_{\substack{l\to\infty\\ k\to\infty}} \langle f\rangle_{I_{-l}(x_1)\times J_{-k}(x_2)} = 0.$$

Since Lebesgue differentiation theorem holds with dyadic rectangles for $f \in L^p$ with $p > 1$, we get

$$\lim_{\substack{l\to\infty\\ k\to\infty}} \langle f\rangle_{I_l(x_1)\times J_k(x_2)} = f(x_1, x_2)$$

for almost every $(x_1, x_2) \in \mathbb{R}^{n+m}$.

It remains to show that II and III tend to zero. We show this only for the term II since the term III can be treated similarly. Now, we have

$$\begin{aligned}|\langle f\rangle_{I_l(x_1)\times J_{-k}(x_2)}| &\le \langle M^1_{\mathcal{D}^n} f(x_1,\cdot)\rangle_{J_{-k}(x_2)}\\ &\le |J_{-k}(x_2)|^{-\frac{1}{p}} \left\|M^1_{\mathcal{D}^n} f(x_1,\cdot)\right\|_{L^p(\mathbb{R}^m)}.\end{aligned}$$

By proof of Lemma 2.4 we have

$$\left\|M^1_{\mathcal{D}^n} f\right\|_{L^p(\mathbb{R}^{n+m})} \lesssim \|f\|_{L^p(\mathbb{R}^{n+m})}.$$

Thus it follows that $\|M^1_{\mathcal{D}^n} f(x_1,\cdot)\|_{L^p(\mathbb{R}^m)} < \infty$ for almost every $x_1 \in \mathbb{R}^n$. Therefore, we get

$$|\langle f\rangle_{I_l(x_1)\times J_{-k}(x_2)}| \le |J_{-k}(x_2)|^{-\frac{1}{p}} \left\|M^1_{\mathcal{D}^n} f(x_1,\cdot)\right\|_{L^p(\mathbb{R}^m)}$$

where the left-hand side tends to zero as $k \to \infty$. □

Lemma 2.16. *Let $\mathcal{D}^n$ and $\mathcal{D}^m$ be dyadic grids. For $p \in (1,\infty)$ and $f \in L^p(\mathbb{R}^{n+m})$ there holds*

$$\lim_{\substack{l\to\infty\\ k\to\infty}} \Bigg\| \sum_{\substack{I\times J\in\mathcal{D}^n\times\mathcal{D}^m\\ 2^{-l}<\ell(I)\le 2^l\\ 2^{-k}<\ell(J)\le 2^k}} \Delta_{I\times J} f - f \Bigg\|_{L^p(\mathbb{R}^{n+m})} = 0.$$

Proof. Continuing from (2.15) we estimate

$$\left|\langle f\rangle_{I_l(x_1)\times J_k(x_2)} - \langle f\rangle_{I_l(x_1)\times J_{-k}(x_2)} - \langle f\rangle_{I_{-l}(x_1)\times J_k(x_2)} + \langle f\rangle_{I_{-l}(x_1)\times J_{-k}(x_2)}\right| \le 4M_{\mathcal{D}} f(x_1,x_2).$$

Then by Lemma 2.4 we know that $\|M_{\mathcal{D}}f\|_{L^p(\mathbb{R}^{n+m})} \lesssim \|f\|_{L^p(\mathbb{R}^{n+m})}$ for $p \in (1,\infty)$. Hence, combined with pointwise convergence we get

$$\lim_{\substack{k\to\infty\\ l\to\infty}} \Bigg\| \sum_{\substack{I\times J\in\mathcal{D}^n\times\mathcal{D}^m\\ 2^{-l}<\ell(I)\le 2^l\\ 2^{-k}<\ell(J)\le 2^k}} \Delta_{I\times J}f - f \Bigg\|_{L^p(\mathbb{R}^n)} = 0$$

by dominated convergence theorem. □

Next, we show that we have similar L^2 identity in the bi-parameter setting as with one-parameter.

Lemma 2.17. *If $f \in L^2(\mathbb{R}^{n+m})$, then*

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{R}^{n+m})} &= \sum_{I\times J\in\mathcal{D}^n\times\mathcal{D}^m} \|\Delta_{I\times J}f\|^2_{L^2(\mathbb{R}^{n+m})}\\ &= \sum_{I\times J\in\mathcal{D}^n\times\mathcal{D}^m} |\langle f, h_I\otimes h_J\rangle|^2.\end{aligned}$$

Proof. Since we have desired equalities in one-parameter, we can just iterate by Fubini's theorem

$$\begin{aligned}\sum_{I\times J\in\mathcal{D}^n\times\mathcal{D}^m} \|\Delta_{I\times J}f\|^2_{L^2(\mathbb{R}^{n+m})} &= \iint_{\mathbb{R}^{n+m}} \sum_{I\times J\in\mathcal{D}^n\times\mathcal{D}^m} |\Delta_{I\times J}f|^2\\ &= \sum_{I\in\mathcal{D}^n}\int_{\mathbb{R}^n}\sum_{J\in\mathcal{D}^m}\int_{\mathbb{R}^m} |\Delta^2_J(\Delta^1_I f)|^2\\ &= \sum_{I\in\mathcal{D}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^m} |\Delta^1_I f|^2\\ &= \int_{\mathbb{R}^m}\sum_{I\in\mathcal{D}^n}\int_{\mathbb{R}^n} |\Delta^1_I f|^2\\ &= \iint_{\mathbb{R}^{n+m}} |f|^2 = \|f\|^2_{\mathbb{R}^{n+m}}.\end{aligned}$$

Further, by observing that

$$\langle h^\eta_R, h^\epsilon_R\rangle = \langle h^\eta_I\otimes h^\eta_J, h^\epsilon_I\otimes h^\epsilon_J\rangle = \langle h^\eta_I, h^\epsilon_I\rangle\langle h^\eta_J, h^\epsilon_J\rangle = \delta_{\eta,\epsilon},$$

we have

$$\int_{\mathbb{R}^{n+m}} |\Delta_R f|^2 = \int_{\mathbb{R}^{n+m}} \Bigg| \sum_{\eta\in\{0,1\}\setminus\{0\}} \langle f, h^\eta_R\rangle h^\eta_R \Bigg|^2 = \sum_{\eta\in\{0,1\}\setminus\{0\}} |\langle f, h^\eta_R\rangle|^2.$$

Hence, we get the desired equality

$$\int_{\mathbb{R}^{n+m}} \sum_{R\in\mathcal{D}} |\Delta_R f|^2 = \sum_{R\in\mathcal{D}} |\langle f, h_R\rangle|^2.$$

□

2.8 One-parameter BMO

For a dyadic grid $\mathcal{D}^n$, $1 \le p < \infty$ and a locally integrable function $b : \mathbb{R}^n \to \mathbb{C}$ we define $\|b\|_{\mathrm{BMO}_p(\mathcal{D}^n)}$ by

$$\|b\|_{\mathrm{BMO}_p(\mathcal{D}^n)} := \sup_{I_0} \left(\frac{1}{|I_0|} \int_{I_0} |b - \langle b\rangle_{I_0}|^p \right)^p$$

where the supremum is taken over all dyadic cubes $I_0 \in \mathcal{D}^n$. Recall the classical John-Nirenberg result

$$\|b\|_{\mathrm{BMO}_p(\mathcal{D}^n)} \sim \|b\|_{\mathrm{BMO}_q(\mathcal{D}^n)}$$

for any $p, q \in [1,\infty)$. Then we say that $b \in \mathrm{BMO}_p(\mathcal{D}^n)$ if $\|b\|_{\mathrm{BMO}_p(\mathcal{D}^n)} < \infty$ for some $p \in (1,\infty)$. If $\|b\|_{\mathrm{BMO}_p(\mathcal{D}^n)} < \infty$ for every dyadic grid $\mathcal{D}^n$ for some $p \in [1,\infty)$, then we say that $b \in \mathrm{BMO}_p(\mathbb{R}^n)$.

From previous definition it is not easy to construct right type of bi-parameter version. Hence, we define

$$\|b\|_{\mathrm{BMO}_{p,*}(\mathcal{D}^n)} := \sup_{I_0} \frac{1}{|I_0|^{\frac{1}{p}}} \left\| \left(\sum_{I\subset I_0} |\langle b, h_I\rangle|^2 \frac{1_I}{|I|} \right)^{\frac{1}{2}} \right\|_{L^p}$$

where the supremum is taken over all dyadic cubes $I_0 \in \mathcal{D}^n$.

Now let $1 < p < \infty$ if we recall (2.12) we see that for fixed $I_0 \in \mathcal{D}^n$

$$\begin{aligned}\int_{I_0} |b - \langle b\rangle_{I_0}|^p &= \|1_{I_0}(b - \langle b\rangle_{I_0})\|_{L^p}^p \\ &\sim \left\| \left(\sum_{I\in\mathcal{D}^n} |\Delta_I \big[1_{I_0}(b - \langle b\rangle_{I_0})\big]|^2 \right)^{\frac{1}{2}} \right\|_{L^p}^p .\end{aligned}$$

Observe that

$$1_{I_0}(b - \langle b\rangle_{I_0}) = \sum_{\substack{\tilde{I}\in\mathcal{D}^n \\ \tilde{I}\subset I_0}} \Delta_{\tilde{I}} b.$$

Then recall that $\Delta_I \Delta_{\tilde{I}} b = 0$ if $I \neq \tilde{I}$ and $\Delta_I \Delta_I b = \Delta_I b$.

Hence, we get

$$\int_{I_0} |b - \langle b \rangle_{I_0}|^p \sim \left\| \Big(\sum_{\substack{I \in \mathcal{D}^n \\ I \subset I_0}} |\Delta_I b|^2 \Big)^{\frac{1}{2}} \right\|_{L^p}^p$$

$$\sim \left\| \Big(\sum_{\substack{I \in \mathcal{D}^n \\ I \subset I_0}} |\langle b, h_I \rangle|^2 \frac{1_I}{|I|} \Big)^{\frac{1}{2}} \right\|_{L^p}^p .$$

Then divide both sides by $|I_0|$ and take supremum over all dyadic cubes I_0, we get

$$\|b\|^p_{\mathrm{BMO}_p(\mathcal{D}^n)} \sim \|b\|^p_{\mathrm{BMO}_{p,*}(\mathcal{D}^n)} .$$

Thus we can use the latter definition instead of the standard definition. Notice that with $p = 2$ we have equality. From the latter definition it is more clear how we define the bi-parameter BMO spaces.

2.9 Bi-parameter BMO

Let $\mathcal{D} = \mathcal{D}^n \times \mathcal{D}^m$ for some dyadic grids $\mathcal{D}^n$ in $\mathbb{R}^n$ and $\mathcal{D}^m$ in $\mathbb{R}^m$. Let $(a_R)_R$ be a complex sequence indexed over dyadic rectangles in $\mathcal{D}$ and $0 < p < \infty$. We say that $(a_R)_R \in \mathrm{BMO}_{rect,p}(\mathcal{D})$ if

$$\|(a_R)_R\|_{\mathrm{BMO}_{rect,p}(\mathcal{D})} := \sup_{R_0} \frac{1}{|R_0|^{\frac{1}{p}}} \left\| \Big(\sum_{R \subset R_0} \frac{|a_R|^2}{|R|} 1_R \Big)^{\frac{1}{2}} \right\|_{L^p} < \infty, \tag{2.18}$$

where the supremum is taken over all possible dyadic rectangles $R_0 \in \mathcal{D}$. Now observe that this definition is a natural analogue of one-parameter $\|\cdot\|_{\mathrm{BMO}^{\mathcal{D}^n}_{p,*}}$.

Furthermore, we say that $(a_R)_R \in \mathrm{BMO}_{prod,p}(\mathcal{D})$ if

$$\|(a_R)_R\|_{\mathrm{BMO}_{prod,p}(\mathcal{D})} := \sup_{\Omega} \frac{1}{|\Omega|^{\frac{1}{p}}} \left\| \Big(\sum_{R \subset \Omega} \frac{|a_R|^2}{|R|} 1_R \Big)^{\frac{1}{2}} \right\|_{L^p} < \infty, \tag{2.19}$$

where the supremum is taken over all possible sets Ω such that for every $x \in \Omega$ there exist $R \in \mathcal{D}$ so that $x \in R \subset \Omega$.

In addition, we denote $\|(a_R)_R\|_{\mathrm{BMO}_{prod,(p,\infty)}}$ by replacing $\|\cdot\|_{L^p}$ with $\|\cdot\|_{L^{p,\infty}}$ in (2.19).

We set

$$\|b\|_{\mathrm{BMO}_{prod,p}(\mathcal{D})} := \|(\langle b, h_R\rangle)_R\|_{\mathrm{BMO}_{prod,p}(\mathcal{D})}$$

for $b \in L^1_{loc}(\mathbb{R}^{n+m})$. If

$$\|b\|_{\mathrm{BMO}_{prod,p}(\mathcal{D})} < \infty$$

for every dyadic grid $\mathcal{D}$ and some $p \in (0, \infty)$, then we say that $b \in \mathrm{BMO}_{prod,p}(\mathbb{R}^n \times \mathbb{R}^m)$. Since we actually consider $p = 2$, we write $\mathrm{BMO}_{prod} = \mathrm{BMO}_{prod,2}$.

To see that the rectangular BMO space is not the correct product BMO space for our purpose, we are going to show that $\mathrm{BMO}_{prod}(\mathcal{D}) \subsetneq \mathrm{BMO}_{rect}(\mathcal{D})$. Then in the last chapter we are going show that the general product BMO space is the necessary condition for boundedness of the bi-parameter paraproducts. These paraproducts are essential for the bi-parameter $T1$ theorems.

Moreover, in Chapter 4 we prove that $T1 \in \mathrm{BMO}_{prod}(\mathbb{R}^n \times \mathbb{R}^m)$, where T is a bi-parameter Calderón-Zygmund operator such that $T : L^2 \to L^2$. In the proof we need the following inequality.

2.10 Minkowski's integral inequality

Lemma 2.20. *(Minkowski's integral inequality) Suppose (X, μ) and (Y, ν) are σ-finite measure spaces and $F : X \times Y \to \mathbb{R}$ is measurable. Then there holds*

$$\left(\int_X \left|\int_Y F(x, y)\,\mathrm{d}\nu(y)\right|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} \le \int_Y \left(\int_X |F(x, y)|^p\,\mathrm{d}\mu(x)\right)^{\frac{1}{p}} \mathrm{d}\nu(y) \tag{2.21}$$

for $1 \le p < \infty$.

Proof. Let (X, μ) and (Y, ν) be σ-finite a measure spaces and let $F : X \times Y \to \mathbb{R}$ be measurable. We denote

$$\int_Y F(x, y)\,\mathrm{d}\nu(y) =: g(x).$$

Now by duality we have

$$\begin{aligned}\left(\int_X\left|\int_Y F(x,y)\,\mathrm{d}\nu(y)\right|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} &= \|g\|_{L^p(\mu)}\\ &= \sup_{\|f\|_{L^{p'}(\mu)}\leq 1}\left|\int_X f(x)g(x)\,\mathrm{d}\mu(x)\right|\\ &\leq \sup_{\|f\|_{L^{p'}(\mu)}\leq 1}\int_X |f(x)|\int_Y |F(x,y)|\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x)\\ &= \sup_{\|f\|_{L^{p'}(\mu)}\leq 1}\int_Y\int_X |f(x)||F(x,y)|\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y).\end{aligned}$$

Then by Hölder's inequality we have

$$\int_X |f(x)||F(x,y)|\,\mathrm{d}\mu(x) \leq \|F(\cdot,y)\|_{L^p(\mu)}\,\|f\|_{L^{p'}(\mu)}\,.$$

Notice that $\|f\|_{L^{p'}(\mu)} \leq 1$ so we obtain

$$\left(\int_X\left|\int_Y F(x,y)\,\mathrm{d}\nu(y)\right|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} \leq \int_Y\left(\int_X |F(x,y)|^p\,\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\mathrm{d}\nu(y).$$

□

Chapter 3

Properties of the product BMO spaces

In this chapter, we first show that John-Nirenberg result holds also in the product space $\mathbb{R}^{n+m}$. After that we show Carleson's counterexample which states that the rectangular BMO space is not contained in the product BMO space. Last subject in this chapter is a useful estimate for sequences involving a product BMO sequence.

3.1 John-Nirenberg result in the product BMO space

Now we show that the analogue of John-Nirenberg inequality holds for the product BMO space. By this theorem any estimate for particular exponent gives similar estimate for the others.

Theorem 3.1. *Let $(a_R)_R$ be a complex sequence indexed over dyadic rectangles in $\mathcal{D} := \mathcal{D}^n \times \mathcal{D}^m$. Then there holds*

$$\|(a_R)_R\|_{\mathrm{BMO}_{prod,p}(\mathcal{D})} \sim \|(a_R)_R\|_{\mathrm{BMO}_{prod,q}(\mathcal{D})} \tag{3.2}$$

for all $0 < p < q < \infty$.

Proof. It is enough to prove

$$\|(a_R)_R\|_{\mathrm{BMO}_{prod,q}(\mathcal{D})} \lesssim \|(a_R)_R\|_{\mathrm{BMO}_{prod,(p,\infty)}(\mathcal{D})}\,, \tag{3.3}$$

for $q > 2$, since by Hölder's inequality we have

$$\begin{aligned}(3.4)\qquad \frac{1}{|\Omega|^{\frac{1}{p}}}\left\| \left(\sum_{R\subset\Omega} \frac{|a_R|^2}{|R|} 1_R \right)^{\frac{1}{2}} \right\|_{L^p} &\leq \frac{1}{|\Omega|^{\frac{1}{p}}} \left(\left(\int_\Omega \left(\sum_{R\subset\Omega} \frac{|a_R|^2}{|R|} 1_R \right)^{\frac{p}{2}\frac{q}{p}} \right)^{\frac{p}{q}} |\Omega|^{1-\frac{p}{q}} \right)^{\frac{1}{p}} \\ &= \frac{1}{|\Omega|^{\frac{1}{q}}}\left\| \left(\sum_{R\subset\Omega} \frac{|a_R|^2}{|R|} 1_R \right)^{\frac{1}{2}} \right\|_{L^q},\end{aligned}$$

for every $0 < p < q < \infty$. Furthermore, the weak-L^p space contains the L^p space, that is, we have the estimate

$$\|(a_R)_R\|_{\mathrm{BMO}_{prod,(p,\infty)}(\mathcal{D})} \leq \|(a_R)_R\|_{\mathrm{BMO}_{prod,p}(\mathcal{D})}.$$

First, let us assume that $a_R \neq 0$ for finitely many R. Thus we have that

$$\|(a_R)_R\|_{\mathrm{BMO}_{prod,q}(\mathcal{D})} < \infty.$$

Then we may fix $\Omega_0 \subset \mathbb{R}^n \times \mathbb{R}^m$ such that

$$(3.5)\qquad \frac{1}{|\Omega_0|^{\frac{1}{q}}}\left\| \left(\sum_{R\subset\Omega_0} \frac{|a_R|^2}{|R|} 1_R \right)^{\frac{1}{2}} \right\|_{L^q} = \|(a_R)_R\|_{\mathrm{BMO}_{prod,q}(\mathcal{D})} =: B.$$

Furthermore, for the right hand side of (3.3) we have

$$(3.6)\qquad \frac{1}{|\Omega_0|^{\frac{1}{p}}}\left\| \left(\sum_{R\subset\Omega_0} \frac{|a_R|^2}{|R|} 1_R \right)^{\frac{1}{2}} \right\|_{L^{p,\infty}} \leq \|(a_R)_R\|_{\mathrm{BMO}_{prod,(p,\infty)}(\mathcal{D})} =: A.$$

For a given constant $C_0 > 0$ we set

$$E := \left\{ \left(\sum_{R\subset\Omega_0} \frac{|a_R|^2}{|R|} 1_R \right)^{\frac{1}{2}} \geq C_0 A \right\}.$$

Now from the inequality (3.6) we get

$$C_0 A |E|^{\frac{1}{p}} \leq A |\Omega_0|^{\frac{1}{p}}.$$

Hence, we have the estimate

$$|E| \leq \frac{|\Omega_0|}{{C_0}^p}.$$

Let us define $\mathcal{R}_1$ and $\mathcal{R}_2$ by

$$\mathcal{R}_1 := \left\{ R : |R \cap E| > \frac{1}{2}|R| \right\}$$

and

$$\mathcal{R}_2 := \left\{ R : |R \cap E| \le \frac{1}{2}|R| \right\}.$$

Now we can estimate B as follows

$$B = \frac{1}{|\Omega_0|^{\frac{1}{q}}} \left\| \left(\sum_{R \subset \Omega_0} \frac{|a_R|^2}{|R|} 1_R \right)^{\frac{1}{2}} \right\|_{L^q} \le \mathrm{I} \ + \mathrm{II}\ , \tag{3.7}$$

where

$$\mathrm{I} \ := \frac{1}{|\Omega_0|^{\frac{1}{q}}} \left\| \left(\sum_{\substack{R \subset \Omega_0 \\ R \in \mathcal{R}_1}} \frac{|a_R|^2}{|R|} 1_R \right)^{\frac{1}{2}} \right\|_{L^q}$$

and

$$\mathrm{II} \ := \frac{1}{|\Omega_0|^{\frac{1}{q}}} \left\| \left(\sum_{\substack{R \subset \Omega_0 \\ R \in \mathcal{R}_2}} \frac{|a_R|^2}{|R|} 1_R \right)^{\frac{1}{2}} \right\|_{L^q}.$$

Now we observe that if $R \in \mathcal{R}_1$, then

$$R \subset \left\{ M_{\mathcal{D}} 1_E > \frac{1}{2} \right\} =: \widetilde{E}.$$

By Chebyshev's inequality and Lemma 2.4, we have that

$$\begin{aligned} |\widetilde{E}| &\le \frac{1}{4} \int_{\mathbb{R}^{n+m}} (M_{\mathcal{D}} 1_E(x_1, x_2))^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\ &\le \frac{C_1}{4} \int_{\mathbb{R}^{n+m}} 1_E(x_1, x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\ &= C_2 |E|. \end{aligned} \tag{3.8}$$

Thus, we have estimate

$$\mathrm{I} \le \frac{1}{|\Omega_0|^{\frac{1}{q}}} \left\| \left(\sum_{R : R \subset \widetilde{E}} \frac{|a_R|^2}{|R|} 1_R \right)^{\frac{1}{2}} \right\|_{L^q} \le |\Omega_0|^{-\frac{1}{q}} B |\widetilde{E}|^{\frac{1}{q}} \le |\Omega_0|^{-\frac{1}{q}} C_2^{\frac{1}{q}} B |E|^{\frac{1}{q}} \le \frac{1}{2} B \tag{3.9}$$

when we choose C_0 large enough so that

$$C_2^{\frac{1}{q}} C_0^{-\frac{p}{q}} \leq \frac{1}{2}.$$

Now we make an estimation for II.[1] Let q^* be such that $2/q + 1/q^* = 1$. Then by duality we have

$$\begin{aligned}
|\Omega_0|^{\frac{1}{q}} \text{ II } &= \Bigg\| \sum_{\substack{R\subset\Omega_0 \\ R\in\mathcal{R}_2}} \frac{|a_R|^2}{|R|} 1_R \Bigg\|_{L^{\frac{q}{2}}}^{\frac{1}{2}} = \Bigg(\sup_{g:\|g\|_{L^{q^*}}\leq 1} \Bigg| \int_{\mathbb{R}^{n+m}} \Bigg(\sum_{\substack{R\subset\Omega_0 \\ R\in\mathcal{R}_2}} \frac{|a_R|^2}{|R|} 1_R \Bigg) g \Bigg| \Bigg)^{\frac{1}{2}} \\
&\leq \Bigg(\sup_{g:\|g\|_{L^{q^*}}\leq 1} \sum_{\substack{R\subset\Omega_0 \\ R\in\mathcal{R}_2}} \frac{|a_R|^2}{|R|} |R| \langle |g| \rangle_R \Bigg)^{\frac{1}{2}}.
\end{aligned}$$

Further, for $R \in \mathcal{R}_2$ we have that $|R \cap E^c| > |R|/2$ hence

$$\begin{aligned}
|\Omega_0|^{\frac{1}{q}} \text{ II } &\leq \Bigg(2 \sup_{g:\|g\|_{L^{q^*}}\leq 1} \sum_{\substack{R\subset\Omega_0 \\ R\in\mathcal{R}_2}} \frac{|a_R|^2}{|R|} |R \cap E^c| \langle |g| \rangle_R \Bigg)^{\frac{1}{2}} \\
&\leq \Bigg(2 \sup_{g:\|g\|_{L^{q^*}}\leq 1} \int_{\Omega_0\setminus E} \sum_{\substack{R\subset\Omega_0 \\ R\in\mathcal{R}_2}} \frac{|a_R|^2}{|R|} 1_R Mg \Bigg)^{\frac{1}{2}} \\
&\leq \Bigg(2 \sup_{g:\|g\|_{L^{q^*}}\leq 1} \|Mg\|_{L^{q^*}} \Bigg\| 1_{\Omega_0\setminus E} \sum_{\substack{R\subset\Omega_0 \\ R\in\mathcal{R}_2}} \frac{|a_R|^2}{|R|} 1_R \Bigg\|_{L^{\frac{q}{2}}} \Bigg)^{\frac{1}{2}} \\
&\leq C_3 \Bigg\| 1_{\Omega_0\setminus E} \Bigg(\sum_{\substack{R\subset\Omega_0 \\ R\in\mathcal{R}_2}} \frac{|a_R|^2}{|R|} 1_R \Bigg)^{\frac{1}{2}} \Bigg\|_{L^q}.
\end{aligned}$$

Then by recalling the definition of E we have that

$$\text{II } \leq |\Omega_0|^{-\frac{1}{q}} C_3 C_0 |\Omega_0 \setminus E|^{\frac{1}{q}} A \leq CA.$$

Now we have achieved the estimation

$$\|(a_R)_R\|_{\mathrm{BMO}_{prod,q}(\mathcal{D})} = B \leq 2CA = 2C \, \|(a_R)_R\|_{\mathrm{BMO}_{prod,(p,\infty)}(\mathcal{D})}$$

whenever we have $a_R \neq 0$ for finitely many R. Hence, the general case follows by monotone convergence theorem. □

[1] In the book by Muscalu and Schlag [6] the estimation for II is done in a different way and in fact there is an estimate that is not correct.

3.2 Carleson's counterexample

To justify two different definitions for the bi-parameter BMO space, we show Carleson's counterexample which proves that the general product BMO space does not coincide with the rectangular one. In other words, we are going to give an example of a sequence in the general product BMO space which does not belong to the rectangular BMO space.

Theorem 3.10. *Let $\mathcal{D}$ be a grid of dyadic rectangles. There holds*

$$\mathrm{BMO}_{rect,2}(\mathcal{D}) \neq \mathrm{BMO}_{prod,2}(\mathcal{D}).$$

Proof. First, we observe that

$$\mathrm{BMO}_{prod,2}(\mathcal{D}) \subset \mathrm{BMO}_{rect,2}(\mathcal{D})$$

holds by the definition. Thus we need to prove that

$$\mathrm{BMO}_{rect,2}(\mathcal{D}) \not\subset \mathrm{BMO}_{prod,2}(\mathcal{D}).$$

For simplicity we assume that product space is $\mathbb{R}\times\mathbb{R}$ and also we assume that $\mathcal{D} = \mathcal{D}^1 \times \mathcal{D}^1$, where $\mathcal{D}^1 = \left\{2^{-k}([0,1)+l) : k \in \mathbb{Z}, l \in \mathbb{Z}\right\}$.

Suppose that a collection of dyadic rectangles $\mathcal{R}$ has the following properties

(1) $R \subset [0,1] \times [0,1]$, for every $R \in \mathcal{R}$;

(2) $\sum_{R\in\mathcal{R}} |R| = 1$; $\qquad (\star)$

(3) $\sum_{\substack{R\in\mathcal{R}\\ R\subset Q}} |R| \leq |Q|$ for every dyadic rectangle $Q \subset [0,1]\times[0,1]$.

Now defining $a_R := |R|^{\frac{1}{2}}$ for $R \in \mathcal{R}$ and $a_R = 0$ otherwise. We have that

$$\frac{1}{|R_0|^{\frac{1}{2}}} \left\| \left(\sum_{R\subset R_0} \frac{|a_R|^2}{|R|} 1_R \right)^{\frac{1}{2}} \right\|_2 \leq \frac{1}{|R_0|^{\frac{1}{2}}} \left(\sum_{\substack{R\subset R_0\\ R\in\mathcal{R}}} |R| \right)^{\frac{1}{2}} \leq 1$$

for an arbitrary dyadic rectangle $R_0 \subset [0,1]\times[0,1]$ by the property (3). On the other hand, if $\Omega := \cup_{R\in\mathcal{R}} R$, then we have that

$$\frac{1}{|\Omega|^{\frac{1}{2}}} \left\| \left(\sum_{R\subset\Omega} \frac{|a_R|^2}{|R|} 1_R \right)^{\frac{1}{2}} \right\|_2 = \frac{1}{|\Omega|^{\frac{1}{2}}} \left(\sum_{R\in\mathcal{R}} |R| \right)^{\frac{1}{2}} = \frac{1}{|\Omega|^{\frac{1}{2}}}$$

by the property (2).

The claim is that we can construct a collection $\mathcal{R}$ such that $|\bigcup_{R\in\mathcal{R}} R|$ is arbitrarily small, which implies that the general product BMO norm of the above sequence is arbitrarily large.

Now we want to show that we can construct a collection $\widetilde{\mathcal{R}}$ with the property $(\star)$ and with area of $\sigma - 1/4\,\sigma^2$ from a collection $\mathcal{R}$ with the property $(\star)$ and area of σ. Continuing this iterative process we obtain a sequence of collections $(\mathcal{R}_k)_k$ with corresponding areas $(\sigma_k)_k$ satisfying the recursion relation

$$\sigma_{k+1} = \sigma_k - \frac{1}{4}\sigma_k^2.$$

We may start iteration with $\mathcal{R}_0 = \{[0,1]\times[0,1]\}$ and area $\sigma_0 = 1$, then the sequence $(\sigma_k)_k$ is decreasing and positive. Hence, the sequence $(\sigma_k)_k$ has a limit L. Moreover, the limit L is zero since

$$L = L - \frac{1}{4}L^2.$$

Let $\mathcal{R}$ be a finite collection of dyadic rectangles with the property $(\star)$ and area of σ. Let N be a positive integer such that

$$\frac{1}{2^N} \le \min\{\ell(I), \ell(J)\}$$

for every $R = I\times J \in \mathcal{R}$. Define the transformations A_j^1 and A_j^2 by

$$A_j^1(x,y) = \left(\frac{j}{2^N} + \frac{x}{2^{N+1}}, y\right)$$

and

$$A_j^2(x,y) = \left(x, \frac{j}{2^N} + \frac{y}{2^{N+1}}\right)$$

for every $j = 0,1,2,\dots,2^N-1$. These are described in Figure 3.1.

Thus we define a new collection $\widetilde{\mathcal{R}}$ by

$$\widetilde{\mathcal{R}} := \bigcup_{i=1}^{2}\bigcup_{j=0}^{2^N-1} A_j^i(\mathcal{R}). \tag{3.11}$$

Now we want to show that $\widetilde{\mathcal{R}}$ satisfies the property $(\star)$. Clearly, we see from the construction that $R\subset[0,1]\times[0,1]$ for every $R\in\widetilde{\mathcal{R}}$.

By the construction, for every $i = 1,2$ and $j = 0,1,2,\dots,2^N-1$ there holds that

$$\sum_{R\in A_j^i(\mathcal{R})} |R| = \frac{1}{2^{N+1}}\sum_{R\in\mathcal{R}} |R| = \frac{1}{2^{N+1}}.$$

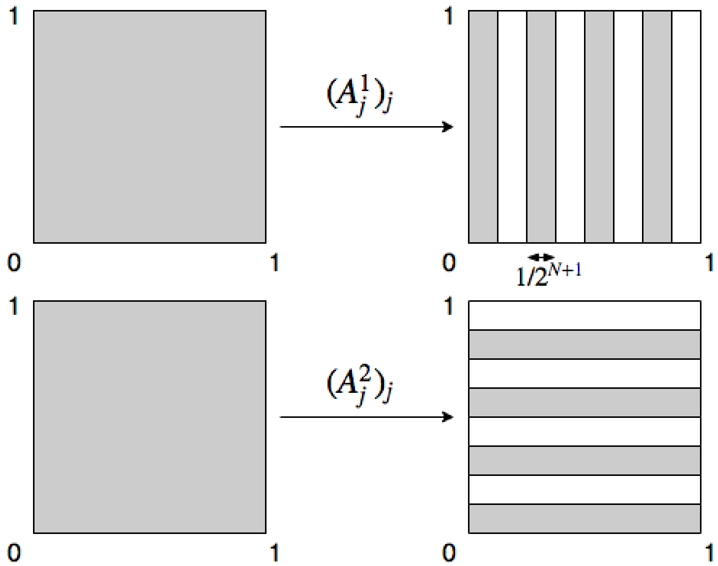

Figure 3.1

Thus, we obtain

$$\sum_{R\in\widetilde{\mathcal{R}}} |R| = \sum_{i=1}^{2}\sum_{j=0}^{2^N-1}\sum_{R\in A_j^i(\mathcal{R})} |R| = 1.$$

Let $Q = I \times J \subset [0,1]\times[0,1]$ be a dyadic rectangle and we show that

$$\sum_{\substack{R\in\widetilde{\mathcal{R}}\\ R\subset Q}} |R| \le |Q|. \tag{3.12}$$

Since Q is dyadic rectangle, we have that $\ell(I) = 1/2^{k_1}$ and $\ell(J) = 1/2^{k_2}$, where k_1 and k_2 are non-negative integers. If $k_1 > N$, then there is at most one column $A_j^1(\mathcal{R})$ for which rectangles are contained in Q. On the other hand, there is no row $A_j^2(\mathcal{R})$ such that rectangles are contained in Q. Now let $\widetilde{Q} = (A_j^1)^{-1}Q$ then by the property (3) of the collection $\mathcal{R}$ we get

$$\sum_{\substack{R\in\widetilde{\mathcal{R}}\\ R\subset Q}} |R| = \sum_{\substack{R\in A_j^1(\mathcal{R})\\ R\subset Q}} |R| = \frac{1}{2^{N+1}}\sum_{\substack{R\in\mathcal{R}\\ R\subset\widetilde{Q}}} |R| \le \frac{1}{2^{N+1}}|\widetilde{Q}| = |Q|.$$

The case $k_2 > N$ can be treated similarly.

Now we consider the case $k_1, k_2 \le N$. Then we observe that Q intersects $2^N/2^{k_1}$ columns $A_j^1(\mathcal{R})$ and $2^N/2^{k_2}$ rows $A_j^2(\mathcal{R})$. Observe that for $R \in A_j^1(\mathcal{R})$ such that $R \subset Q$

we know that $R \subset [j/2^N, j/2^N + 1/2^{N+1}] \times J$. Hence, by the previous case that $k_1 > N$ we have that

$$\sum_{\substack{R \in A_j^1(\mathcal{R}) \\ R \subset Q}} |R| \leq \frac{1}{2^{N+1}} |J|$$

for each j such that the column $A_j^1(\mathcal{R})$ intersects the rectangle Q. By similar argument each $A_j^2(\mathcal{R})$ contributes at most $1/2^{N+1}|I|$ to the summation. Hence, we have that

$$\begin{aligned}
\sum_{\substack{R \in \widetilde{\mathcal{R}} \\ R \subset Q}} |R| &= \sum_{\substack{R \in \bigcup_j A_j^1(\mathcal{R}) \\ R \subset Q}} |R| + \sum_{\substack{R \in \bigcup_j A_j^2(\mathcal{R}) \\ R \subset Q}} |R| \\
&\leq \frac{2^N}{2^{k_1}} \frac{1}{2^{N+1}} |J| + \frac{2^N}{2^{k_2}} \frac{1}{2^{N+1}} |I| \\
&= \frac{|I||J|}{2} + \frac{|I||J|}{2} = |Q|.
\end{aligned} \tag{3.13}$$

Now, since we confirmed that $\widetilde{\mathcal{R}}$ has the property $(\star)$, we need to prove that

$$\left| \bigcup_{R \in \widetilde{\mathcal{R}}} R \right| = \sigma - \frac{1}{4}\sigma^2.$$

By the construction for each $i = 1, 2$ we have that

$$\left| \bigcup_{j=0}^{2^N-1} A_j^i(\mathcal{R}) \right| = \sum_{j=0}^{2^N-1} |A_j^i(\mathcal{R})| = \sum_{j=0}^{2^N-1} \left| \bigcup_{R \in A_j^i(\mathcal{R})} R \right| = \sum_{j=0}^{2^N-1} \frac{\sigma}{2^{N+1}} = \frac{1}{2}\sigma.$$

Thus we obtain

$$\begin{aligned}
\left| \bigcup_{R \in \widetilde{\mathcal{R}}} R \right| &= \left| \left(\bigcup_{j_1=0}^{2^N-1} A_{j_1}^1(\mathcal{R}) \right) \cup \left(\bigcup_{j_2=0}^{2^N-1} A_{j_2}^2(\mathcal{R}) \right) \right| \\
&= \left| \bigcup_{j_1=0}^{2^N-1} A_{j_1}^1(\mathcal{R}) \right| + \left| \bigcup_{j_2=0}^{2^N-1} A_{j_2}^2(\mathcal{R}) \right| - \left| \left(\bigcup_{j_1=0}^{2^N-1} A_{j_1}^1(\mathcal{R}) \right) \cap \left(\bigcup_{j_2=0}^{2^N-1} A_{j_2}^2(\mathcal{R}) \right) \right| \\
&= \sigma - \left| \left(\bigcup_{j_1=0}^{2^N-1} A_{j_1}^1(\mathcal{R}) \right) \cap \left(\bigcup_{j_2=0}^{2^N-1} A_{j_2}^2(\mathcal{R}) \right) \right|.
\end{aligned} \tag{3.14}$$

By disjointness we can decompose the last term to

$$\sum_{j_1, j_2=0}^{2^N-1} \left| \left(I_{j_1} \times I_{j_2} \cap \bigcup_{R \in A_{j_1}^1(\mathcal{R})} R \right) \cap \left(I_{j_1} \times I_{j_2} \cap \bigcup_{R \in A_{j_2}^2(\mathcal{R})} R \right) \right|,$$

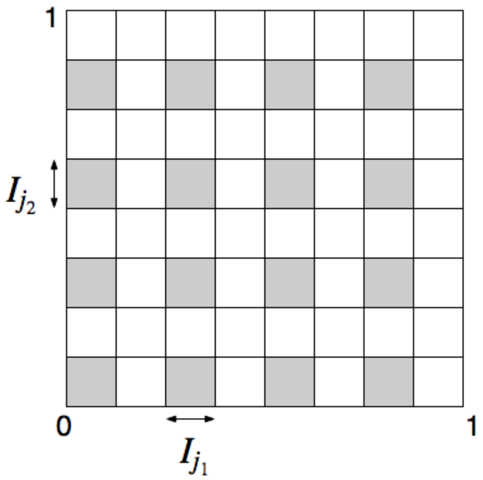

Figure 3.2

where $I_j = [j/2^N, j/2^N + 1/2^{N+1}]$ as shown in Figure 3.2. Thus, we write

$$I_{j_1} \times I_{j_2} \cap \bigcup_{R \in A^1_{j_1}(\mathcal{R})} R = E_{j_1,j_2} \times I_{j_2}$$

and

$$I_{j_1} \times I_{j_2} \cap \bigcup_{R \in A^2_{j_2}(\mathcal{R})} R = I_{j_1} \times F_{j_1,j_2},$$

where

$$E_{j_1,j_2} = I_{j_1} \cap \bigcup_{\substack{I_R \times J_R \in A^1_{j_1}(\mathcal{R}) \\ J_R \cap I_{j_2} \neq \emptyset}} I_R$$

and

$$F_{j_1,j_2} = I_{j_2} \cap \bigcup_{\substack{I_R \times J_R \in A^2_{j_2}(\mathcal{R}) \\ I_R \cap I_{j_1} \neq \emptyset}} J_R.$$

Illustration of the sets E_{j_1,j_2} and F_{j_1,j_2} is shown in Figure 3.3 for fixed $I_{j_1} \times I_{j_2}$. Hence, we have that

$$|(E_{j_1,j_2} \times J_{j_2}) \cap (I_{j_1} \times F_{j_1,j_2})| = |E_{j_1,j_2} \times F_{j_1,j_2}| = |E_{j_1,j_2}||F_{j_1,j_2}|.$$

Since the transformation $A^1_{j_1}$ is symmetrical for each j_1, it follows that the measure of E_{j_1,j_2} does not depend on j_1 and with same deduction $|F_{j_1,j_2}|$ does not depend on j_2.

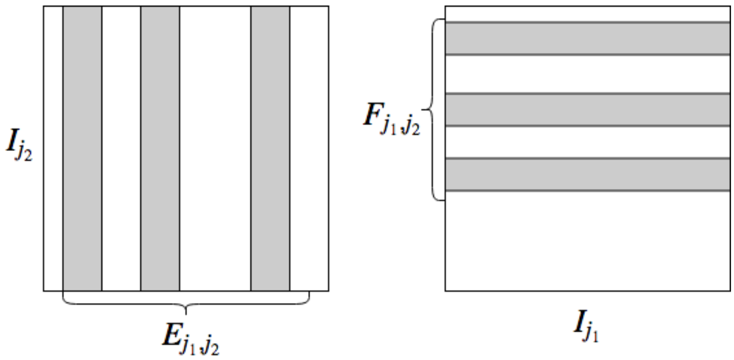

Figure 3.3

Hence, we denote $|E_{j_1,j_2}|$ by $|E_{j_2}|$ and $|F_{j_1,j_2}|$ by $|F_{j_1}|$. Thus we have that

$$\left|\left(\bigcup_{j_1=0}^{2^N-1} A^1_{j_1}(\mathcal{R})\right) \cap \left(\bigcup_{j_2=0}^{2^N-1} A^2_{j_2}(\mathcal{R})\right)\right| = \sum_{j_2=0}^{2^N-1} |E_{j_2}| \sum_{j_1=0}^{2^N-1} |F_{j_1}|. \tag{3.15}$$

Furthermore, by the definition of the transformation for every j_1 we have

$$\bigcup_{R \in A^1_{j_1}(\mathcal{R})} R \subset \left[\frac{j_1}{2^N}, \frac{j_1}{2^N} + \frac{1}{2^{N+1}}\right] \times [0,1].$$

Then for every $R = I \times J \in A^1_{j_1}(\mathcal{R})$ we can write

$$R = \bigcup_{j_2=0}^{2^N-1} I \times \left(J \cap \left[\frac{j_2}{2^N}, \frac{j_2+1}{2^N}\right]\right).$$

By the assumption that $\ell(J) \geq 1/2^N$ we know that if $J \cap \left[j_2/2^N, (j_2+1)/2^N\right] \neq \emptyset$, then there exists E_{j_1,j_2} such that $I \subset E_{j_1,j_2}$. Hence, we have

$$\bigcup_{R \in A^1_{j_1}(\mathcal{R})} R \subset \bigcup_{j_2=0}^{2^N-1} E_{j_1,j_2} \times \left[\frac{j_2}{2^N}, \frac{j_2+1}{2^N}\right].$$

Recall that

$$E_{j_1,j_2} = \bigcup_{\substack{I \times J \in A^1_{j_1}(\mathcal{R}) \\ J \cap I_{j_2} \neq \emptyset}} I.$$

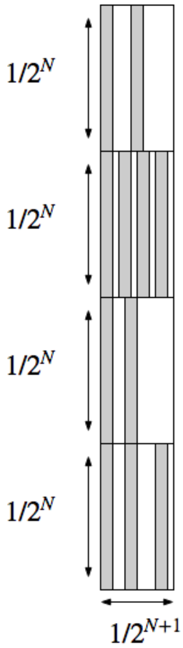

Figure 3.4

Hence, by the fact that each $R = I \times J \in A^1_{j_1}(\mathcal{R})$ is a dyadic rectangle with $\ell(J) \geq 1/2^N$ we have that

$$\bigcup_{j_2=0}^{2^N-1} E_{j_1,j_2} \times \left(\frac{j_2}{2^N}, \frac{j_2+1}{2^N}\right) \subset \bigcup_{R \in A^1_{j_1}(\mathcal{R})} R.$$

Description of the rectangles of $A^1_{j_1}(\mathcal{R})$ is shown in Figure 3.4.

Thus for every given $A^1_j(\mathcal{R})$ there holds

$$\sum_{j_2=0}^{2^N-1} |E_{j_2}| \frac{1}{2^N} = \left| \bigcup_{R \in A^1_j(\mathcal{R})} R \right| = \frac{\sigma}{2^{N+1}}$$

and with similar argument for every given $A^2_j(\mathcal{R})$ we have

$$\sum_{j_1=0}^{2^N-1} |F_{j_1}| \frac{1}{2^N} = \left| \bigcup_{R \in A^2_j(\mathcal{R})} R \right| = \frac{\sigma}{2^{N+1}}.$$

Substituting these to equation (3.15) we get

$$\left|\left(\bigcup_{j_1=0}^{2^N-1} A^1_{j_1}(\mathcal{R})\right)\cap\left(\bigcup_{j_2=0}^{2^N-1} A^2_{j_2}(\mathcal{R})\right)\right| = \frac{1}{4}\sigma^2.$$

Hence, we obtain

$$\left|\bigcup_{R\in\widetilde{\mathcal{R}}} R\right| = \sigma - \frac{1}{4}\sigma^2. \tag{3.16}$$

Now let $(\sigma_k)_{k\in\mathbb{N}_0}$ be a sequence such that $\sigma_0 = 1$ and

$$\sigma_{k+1} = \sigma_k - \frac{1}{4}\sigma_k^2$$

for $k \geq 1$. Thus we have proved that we can construct a collection $\widetilde{\mathcal{R}}$ with the property $(\star)$ and area of σ_k for every $k \in \mathbb{N}_0$. By the same construction we can construct a collection $\mathcal{R}_k$ in $[k, k+1] \times [0,1]$ for every $k \in \mathbb{N}_0$ with the property $(\star)$ where (1) is modified to hold in $[k, k+1] \times [0,1]$. Further, we can demand that $|\bigcup_{R\in\mathcal{R}_k} R| = \sigma_k$ for every $k \in \mathbb{N}_0$.

Let $\mathcal{R} = \bigcup_{k\in\mathbb{N}_0} \mathcal{R}_k$. Then we define $a_R := |R|^{\frac{1}{2}}$ for $R \in \mathcal{R}$ and $a_R = 0$ otherwise. If $\Omega_k := \bigcup_{R\in\mathcal{R}_k} R$, then we get that

$$\frac{1}{|\Omega_k|^{\frac{1}{2}}}\left\|\left(\sum_{R\subset\Omega_k}\frac{|a_R|^2}{|R|}1_R\right)^{\frac{1}{2}}\right\|_2 \geq \frac{1}{|\Omega_k|^{\frac{1}{2}}}\left\|\left(\sum_{R\in\mathcal{R}_k}\frac{|a_R|^2}{|R|}1_R\right)^{\frac{1}{2}}\right\|_2 = \sigma_k^{-\frac{1}{2}}$$

for every $k \in \mathbb{N}_0$. Hence, we have $\|(a_R)_R\|_{\mathrm{BMO}_{prod,2}(\mathcal{D})} = \infty$. On the other hand, for every rectangle $R_0 \subset \bigcup_{k\in\mathbb{N}_0}[k, k+1] \times [0,1]$ we have that

$$\frac{1}{|R_0|^{\frac{1}{2}}}\left\|\left(\sum_{R\subset R_0}\frac{|a_R|^2}{|R|}1_R\right)^{\frac{1}{2}}\right\|_2 \leq \frac{1}{|R_0|^{\frac{1}{2}}}\left(\sum_{\substack{R\subset R_0\\ R\in\mathcal{R}}}|R|\right)^{\frac{1}{2}} \leq 1.$$

□

3.3 Lemma for BMO sequences

Next, we show useful lemma for proving boundedness of the bi-parameter paraproducts in the last chapter.

Lemma 3.17. *If* $(\lambda_R)_{R\in\mathcal{D}}$ *is a complex sequence such that*

$$\|\lambda\|_{\mathrm{BMO}_{prod}(\mathcal{D})} < \infty,$$

then for all sequences $(A_R)_{R\in\mathcal{D}}$

$$\sum_{R\in\mathcal{D}} |\lambda_R||A_R| \lesssim \|\lambda\|_{\mathrm{BMO}_{prod}(\mathcal{D})} \left\| \left(\sum_{R\in\mathcal{D}} |A_R|^2 \frac{1_R}{|R|} \right)^{\frac{1}{2}} \right\|_{L^1}. \tag{3.18}$$

Proof. Let us denote

$$s_A := \left(\sum_{R\in\mathcal{D}} |A_R|^2 \frac{1_R}{|R|} \right)^{\frac{1}{2}}.$$

We may assume that $\|s_A\|_{L^1} < \infty$. Let $U_k := \left\{ s_A > 2^k \right\}$, $\widetilde{U}_k := \{M1_{U_k} > 1/2\}$ and $\mathcal{R}_k := \{R : |R\cap U_k| > |R|/2\}$. Observe that for $(x_1, x_2) \in R \in \mathcal{R}_k$ we have

$$M1_{U_k}(x_1, x_2) \geq \frac{|R\cap U_k|}{|R|} > \frac{1}{2}$$

implying that $\bigcup_{R\in\mathcal{R}_k} R \subset \widetilde{U}_k$.

Moreover, if $A_R \neq 0$, then

$$s_A(x) \geq \frac{|A_R|}{|R|^{\frac{1}{2}}}$$

for all $x \in R$. Thus $R \subset \left\{ s_A \geq |A_R||R|^{-\frac{1}{2}} \right\} \subset \left\{ s_A > 2^k \right\} =: U_k$ whenever $2^k < |A_R||R|^{-\frac{1}{2}}$, that is, $R \in \bigcup_{R'\in\mathcal{R}_k} R'$ whenever $A_R \neq 0$.

If $R \in \bigcap_{k\in\mathbb{Z}} \mathcal{R}_k$, then $0 = |R\cap\{s_A = \infty\}| \geq |R|/2$ but there is no rectangle R with $|R| = 0$.

Now we can write

$$\sum_{R\in\mathcal{D}} |\lambda_R||A_R| = \sum_{k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}} |\lambda_R||A_R| \leq \sum_{k\in\mathbb{Z}} \left(\sum_{R\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}} |\lambda_R|^2 \right)^{\frac{1}{2}} \left(\sum_{R\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}} |A_R|^2 \right)^{\frac{1}{2}}.$$

Then we can estimate

$$\left(\sum_{R\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}} |\lambda_R|^2 \right)^{\frac{1}{2}} \leq \left(\frac{|\widetilde{U}_k|}{|\widetilde{U}_k|} \sum_{R\subset\widetilde{U}_k} |\lambda_R|^2 \right)^{\frac{1}{2}} \leq |\widetilde{U}_k|^{\frac{1}{2}} \|\lambda\|_{\mathrm{BMO}_{prod}(\mathcal{D})}$$

and

$$\begin{aligned}
\Big(\sum_{R\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}}|A_R|^2\frac{|R|}{|R|}\Big)^{\frac{1}{2}} &\le \Big(2\sum_{R\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}}|A_R|^2\frac{|R\cap U_{k+1}^c|}{|R|}\Big)^{\frac{1}{2}}\\
&\lesssim \Big(\int_{U_{k+1}^c}\sum_{R\in\mathcal{R}_k}|A_R|^2\frac{1_R}{|R|}\Big)^{\frac{1}{2}}\\
&\le \Big(\int_{\widetilde{U}_k\setminus U_{k+1}} s_A^2\Big)^{\frac{1}{2}}\\
&\lesssim 2^k|\widetilde{U}_k|^{\frac{1}{2}}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\sum_{R\in\mathcal{D}}|\lambda_R||A_R| &\lesssim \|\lambda\|_{\mathrm{BMO}_{prod}}\sum_{k\in\mathbb{Z}}2^k|\widetilde{U}_k|\\
&\lesssim \|\lambda\|_{\mathrm{BMO}_{prod}}\sum_{k\in\mathbb{Z}}2^k|U_k|\\
&\sim \|\lambda\|_{\mathrm{BMO}_{prod}}\left\|s_A\right\|_{L^1},
\end{aligned}$$

where $\sum_{k\in\mathbb{Z}}2^k|U_k|\sim\left\|s_A\right\|_{L^1}$ is well-known result from the basic course of real analysis. $\square$

Chapter 4

Bi-parameter Calderón-Zygmund operators

The aim in this chapter is to show $T1 \in \mathrm{BMO}_{prod}(\mathbb{R}^{n+m})$, where T is a bi-parameter Calderón-Zygmund operator such that $T : L^2 \to L^2$. For this theorem we need Journé's covering lemma. Naturally, we need to first define one-parameter Calderón-Zygmund operators and show $L^\infty \to \mathrm{BMO}$ boundedness.

4.1 One-parameter singular integrals

Let $\alpha \in (0, 1]$. We say that U_1 is a Calderón-Zygmund operator on $\mathbb{R}^m$, if the following properties holds:

(1) U_1 is linear.

(2) $U_1 : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^m)$

(3) There exists a kernel $K_{U_1} : \mathbb{R}^m \times \mathbb{R}^m \setminus \{(x_2, y_2) \in \mathbb{R}^m \times \mathbb{R}^m : x_2 = y_2\} \to \mathbb{R}$ associated to the operator U_1 such that

$$\langle U_1 f, g\rangle = \iint_{\mathbb{R}^{2m}} K_{U_1}(x_2, y_2) f(y_2) g(x_2) \,\mathrm{d}x_2 \,\mathrm{d}y_2$$

whenever $\operatorname{spt} f \cap \operatorname{spt} g = \emptyset$.

Moreover, the kernel K_{U_1} satisfies the following size estimate and α-Hölder estimates for some constant $C > 0$.

(3.a)
$$|K_{U_1}(x_2, y_2)| \leq C \frac{1}{|x_2 - y_2|^m}$$

for $x_2 \neq y_2$

(3.b)
$$|K_{U_1}(x_2, y_2) - K_{U_1}(x_2', y_2)| \leq C \frac{|x_2 - x_2'|^\alpha}{|x_2 - y_2|^{m+\alpha}}$$
whenever $x_2 \neq y_2$ and $|x_2 - x_2'| \leq |x_2 - y_2|/2$

(3.c)
$$|K_{U_1}(x_2, y_2) - K_{U_1}(x_2, y_2')| \leq C \frac{|y_2 - y_2'|^\alpha}{|x_2 - y_2|^{m+\alpha}}$$
whenever $x_2 \neq y_2$ and $|y_2 - y_2'| \leq |x_2 - y_2|/2$

For a Calderón-Zygmund operator U we denote
$$\|U\| := \|U\|_{L^2 \to L^2} + \|U\|_{CZ_\alpha},$$
where $\|U\|_{CZ_\alpha}$ denotes the best constant C in the above size and α-Hölder estimates.

Since the indicator function 1 is not an L^2 function, we need to define the pairing $\langle U_1 1, h_J \rangle$, where U_1 is a Calderón-Zygmund operator on $\mathbb{R}^m$. We set
$$\langle U_1 1, h_J \rangle = \langle U_1 1_{3J}, h_J \rangle + \langle U_1 1_{(3J)^c}, h_J \rangle,$$
where
$$\langle U_1 1_{(3J)^c}, h_J \rangle := \int_{(3J)^c} \int_J (K_{U_1}(x_2, y_2) - K_{U_1}(c_J, y_2)) h_J(x_2) \, \mathrm{d}x_2 \, \mathrm{d}y_2$$
is well-defined by the α-Hölder estimate of the kernel K_{U_1}. This will be shown in the next theorem.

In addition, the number 3 has no particular reason in defining the pairings. For example
$$\begin{aligned}
\langle U_1 1_{5J}, h_J \rangle &+ \int_{(5J)^c} \int_J [K_{U_1}(x_2, y_2) - K_{U_1}(c_J, y_2)] h_J(x_2) \, \mathrm{d}x_2 \, \mathrm{d}y_2 \\
&= \langle U_1 1_{5J \setminus 3J}, h_J \rangle + \langle U_1 1_{3J}, h_J \rangle \\
&\quad + \int_{(5J)^c} \int_J [K_{U_1}(x_2, y_2) - K_{U_1}(c_J, y_2)] h_J(x_2) \, \mathrm{d}x_2 \, \mathrm{d}y_2.
\end{aligned}$$
Since the non-cancelative Haar functions have zero mean, we have
$$\langle U_1 1_{5J \setminus 3J}, h_J \rangle = \int_{5J \setminus 3J} \int_J [K_{U_1}(x_2, y_2) - K_{U_1}(c_J, y_2)] h_J(x_2) \, \mathrm{d}x_2 \, \mathrm{d}y_2.$$

Then continue from above we get

$$\begin{aligned}
&\langle U_1 1_{3J}, h_J\rangle + \int_{5J\setminus 3J}\int_J [K_{U_1}(x_2,y_2) - K_{U_1}(c_J,y_2)]h_J(x_2)\,\mathrm{d}x_2\,\mathrm{d}y_2 \\
&\qquad + \int_{(5J)^c}\int_J [K_{U_1}(x_2,y_2) - K_{U_1}(c_J,y_2)]h_J(x_2)\,\mathrm{d}x_2\,\mathrm{d}y_2 \\
&\qquad = \langle U_1 1_{3J}, h_J\rangle + \int_{(3J)^c}\int_J [K_{U_1}(x_2,y_2) - K_{U_1}(c_J,y_2)]h_J(x_2)\,\mathrm{d}x_2\,\mathrm{d}y_2 \\
&\qquad =: \langle U_1 1, h_J\rangle.
\end{aligned}$$

Moreover, in place of the indicator function 1, we can use any L^∞ function and in place of the Haar function h_J we can have any bounded zero mean function supported in J.

Since we defined $U_1 1$ as a pairing, we define that $U_1 1 \in \mathrm{BMO}(\mathbb{R}^m)$ if

$$\sum_{\substack{J\in\mathcal{D}^m \\ J\subset J_0}} |\langle U_1 1, h_J\rangle|^2 \lesssim |J_0|$$

for every dyadic grid $\mathcal{D}^m$ and for every $J_0 \subset \mathcal{D}^m$. Now for $b \in L^1_{loc}(\mathbb{R}^m)$ if

$$|\langle b, a\rangle| \le C|J_0|$$

for all cubes $J_0 \subset \mathbb{R}^m$ and for every $a : \mathbb{R}^m \to \mathbb{C}$ such that $\|a\|_{L^\infty} \le 1$, $\operatorname{spt} a \subset J_0$ and a has mean zero, then $b \in \mathrm{BMO}_1(\mathbb{R}^m)$. Then by the classical John-Nirenberg result we know that $b \in \mathrm{BMO}_2(\mathbb{R}^m)$. Hence, recall discussion in the chapter 2 that then

$$\|b\|_{\mathrm{BMO}_{2,*}(\mathcal{D}^m)} < \infty$$

for all dyadic grids $\mathcal{D}^m$. While $U_1 1$ is not really an L^1_{loc} function, it follows by essentially by the same argument that, $U_1 1 \in \mathrm{BMO}(\mathbb{R}^m)$ if

$$|\langle U_1 1, a\rangle| \lesssim \|U_1\||J_0|,$$

where a is defined as above.

Theorem 4.1. *Let U_1 be a Calderón-Zygmund opertor on $\mathbb{R}^m$ as above defined. Then for every $f \in L^\infty$ and every dyadic grid $\mathcal{D}^m$*

$$\|U_1 f\|_{\mathrm{BMO}(\mathcal{D}^m)} \lesssim \|U_1\|\,\|f\|_{L^\infty}. \tag{4.2}$$

Proof. Let J_0 be a cube on $\mathbb{R}^m$. Fix function a such that $a1_{J_0} = a$,

$$\int a = 0$$

and $\|a\|_{L^\infty} \leq 1$. We assume that $f \equiv 1$ since the general case $f \in L^\infty$ can be proven similarly. Now by observations previously it is enough prove that

$$|\langle U_1 1, a\rangle| \lesssim \|U_1\||J_0|.$$

We write

$$|\langle U_1 1, a\rangle| = |\langle U_1 1_{3J_0}, a\rangle| + |\langle U_1 1_{(3J_0)^c}, a\rangle|$$

and observe that

$$|\langle U_1 1_{3J_0}, a\rangle| \leq \|a\|_{L^\infty} \int_{3J_0} |U_1 1_{3J_0}| \lesssim |J_0|^{\frac{1}{2}} \|U_1 1_{3J_0}\|_{L^2} \lesssim \|U_1\||J_0|^{\frac{1}{2}} \|1_{3J_0}\|_{L^2} \lesssim \|U_1\||J_0|.$$

Thus we need to show that

$$|\langle U_1 1_{(3J_0)^c}, a\rangle| \lesssim |J_0|.$$

Hence, we have

$$\begin{aligned}
|\langle U_1 1_{(3J_0)^c}, a\rangle| &= \left| \int_{(3J_0)^c} \int_{J_0} (K_{U_1}(x_2, y_2) - K_{U_1}(c_{J_0}, y_2)) a(x_2) \, \mathrm{d}x_2 \, \mathrm{d}y_2 \right| \\
&\leq \int_{(3J_0)^c} \int_{J_0} \left| K_{U_1}(x_2, y_2) - K_{U_1}(c_{J_0}, y_2) \right| |a(x_2)| \, \mathrm{d}x_2 \, \mathrm{d}y_2 \\
&\leq \|U_1\| \, \|a\|_{L^\infty} \int_{(3J_0)^c} \int_{J_0} \frac{\ell(J_0)^\alpha}{|c_{J_0} - y_2|^{m+\alpha}} \, \mathrm{d}x_2 \, \mathrm{d}y_2 \\
&\leq \|U_1\||J_0| \int_{(3J_0)^c} \frac{\ell(J_0)^\alpha}{|c_{J_0} - y_2|^{m+\alpha}} \, \mathrm{d}y_2.
\end{aligned}$$

Then we can compute the integral by summing over the annuli

$$A(j) := B(c_{J_0}, \ell(J_0) 2^{j+1}) \setminus B(c_{J_0}, \ell(J_0) 2^j).$$

Since $\ell(J_0)2^j \leq |c_{J_0} - y_2| \leq \ell(J_0)2^{j+1}$, we get

$$\begin{aligned}\int_{(3J_0)^c} \frac{\ell(J_0)^\alpha}{|c_{J_0} - y_2|^{m+\alpha}} \, \mathrm{d}y_2 &\leq \sum_{j=0}^{\infty} \int_{A(j)} \frac{\ell(J_0)^\alpha}{|c_{J_0} - y_2|^{m+\alpha}} \, \mathrm{d}y_2 \\ &\leq \sum_{j=0}^{\infty} \frac{\ell(J_0)^\alpha}{(\ell(J_0)2^j)^{m+\alpha}} |B(c_{J_0}, \ell(J_0)2^{j+1})| \\ &\lesssim \frac{1}{\ell(J_0)^m} \sum_{j=0}^{\infty} \ell(J_0)^m 2^{-j(m+\alpha)} 2^{jm} \\ &= \sum_{j=0}^{\infty} 2^{-j\alpha} \lesssim 1.\end{aligned}$$

Now we can conclude that

$$|\langle U_1 1_{(3J_0)^c}, a \rangle| \lesssim \|U_1\| |J_0|$$

as desired. $\square$

We are going to define the bi-parameter Calderón-Zygmund operator and then show similar result in the product space. Here we got the result directly by calculating but as we are going to see the situation is completely different with the bi-parameter setting.

4.2 Bi-parameter singular integrals

We say that T is a bi-parameter Calderón-Zygmund operator, if the following properties holds:

(1) T is linear.

(2) $T : L^2(\mathbb{R}^{n+m}) \to L^2(\mathbb{R}^{n+m})$

(3) For every $x_1, y_1 \in \mathbb{R}^n$ there exists a Calderón-Zygmund operator $U_1(x_1, y_1)$ on $\mathbb{R}^m$ such that

(3.a)

$$||U_1(x_1, y_1)|| \lesssim \frac{1}{|x_1 - y_1|^n}$$

for $x_1 \neq y_1$,

(3.b)
$$||U_1(x_1,y_1) - U_1(x_1',y_1)|| \lesssim \frac{|x_1 - x_1'|^\alpha}{|x_1 - y_1|^{n+\alpha}}$$
whenever $|x_1 - x_1'| \le |x_1 - y_1|$ and

(3.c)
$$||U_1(x_1,y_1) - U_1(x_1,y_1')|| \lesssim \frac{|y_1 - y_1'|^\alpha}{|x_1 - y_1|^{n+\alpha}}$$
whenever $|y_1 - y_1'| \le |x_1 - y_1|$.

(4) For every $x_2, y_2 \in \mathbb{R}^m$ there exists a Calderón-Zygmund operator $U_2(x_2,y_2)$ on $\mathbb{R}^n$ such that we have symmetrical estimates as for the U_1.

(5) If $\operatorname{spt}_{\mathbb{R}^n} f_1 \cap \operatorname{spt}_{\mathbb{R}^n} f_2 = \emptyset$, then we have representation
$$\langle Tf_1, f_2\rangle = \iint_{\mathbb{R}^{2n}} \langle U_1(x_1,y_1)(f_1(y_1,\cdot)), f_2(\cdot,x_2)\rangle \,\mathrm{d}x_1\,\mathrm{d}y_1$$
where $U_1(x_1,y_1)$ is defined above.

(6) If $\operatorname{spt}_{\mathbb{R}^m} f_1 \cap \operatorname{spt}_{\mathbb{R}^m} f_2 = \emptyset$, then similarly as above
$$\langle Tf_1, f_2\rangle = \iint_{\mathbb{R}^{2m}} \langle U_2(x_2,y_2)(f_1(\cdot,y_2)), f_2(x_1,\cdot)\rangle \,\mathrm{d}x_2\,\mathrm{d}y_2.$$

Let T be a bi-parameter Calderón-Zygmund operator on $\mathbb{R}^{n+m}$. Let $R = I \times J \subset \mathbb{R}^{n+m}$ and $h_R := h_I \otimes h_J$. We define
$$\begin{aligned}\langle T1, h_R\rangle :=&\\ &\langle T(1_{3I}\otimes 1_{3J}), h_R\rangle + \langle T(1_{3I}\otimes 1_{(3J)^c}), h_R\rangle \\ &+ \langle T(1_{(3I)^c}\otimes 1_{3J}), h_R\rangle + \langle T(1_{(3I)^c}\otimes 1_{(3J)^c}), h_R\rangle.\end{aligned}$$
Then we need to define each of the terms properly. We see that the first term is already well-defined.

For the second term we set
$$\langle T(1_{3I}\otimes 1_{(3J)^c}), h_R\rangle := \int_{(3J)^c}\int_J \langle [U_2(x_2,y_2) - U_2(c_J,y_2)]1_{3I}, h_I\rangle h_J(x_2)\,\mathrm{d}x_2\,\mathrm{d}y_2.$$
Since $x_2 \in J$ and $y_2 \in (3J)^c$, we have
$$\begin{aligned}|\langle [U_2(x_2,y_2) - U_2(c_J,y_2)]1_{3I}, h_I\rangle| &\le \|U_2(x_2,y_2) - U_2(c_J,y_2)\|_{L^2\to L^2}\,\|1_{3I}\|_{L^2(\mathbb{R}^n)}\,\|h_I\|_{L^2(\mathbb{R}^n)} \\ &\lesssim |I|^{\frac{1}{2}}\frac{\ell(J)^\alpha}{|y_2 - c_I|^{m+\alpha}}.\end{aligned}$$

Thus we have

$$
\begin{aligned}
|\langle T(1_{3I}\otimes 1_{(3J)^c}),h_R\rangle| &\le \int_{(3J)^c}\int_J |\langle [U_2(x_2,y_2)-U_2(c_J,y_2)]1_{3I},h_I\rangle h_J(x_2)|\,\mathrm{d}x_2\,\mathrm{d}y_2\\
&\lesssim \int_{(3J)^c} |J|^{\frac12}|I|^{\frac12}\frac{\ell(J)^\alpha}{|y_2-c_I|^{m+\alpha}}\,\mathrm{d}y_2\\
&= |I\times J|^{\frac12}\int_{(3J)^c}\frac{\ell(J)^\alpha}{|y_2-c_I|^{m+\alpha}}\,\mathrm{d}y_2\\
&\lesssim |I\times J|^{\frac12}<\infty.
\end{aligned}
$$

For the third term we set

$$
\langle T(1_{(3I)^c}\otimes 1_{3J}),h_R\rangle := \int_{(3I)^c}\int_I \langle [U_1(x_1,y_1)-U_1(c_I,y_1)]1_{3J},h_J\rangle h_I(x_1)\,\mathrm{d}x_1\,\mathrm{d}y_1
$$

and it is bounded with same argument as for the second term.

For the last term we set

$$
\langle T(1_{(3I)^c}\otimes 1_{(3J)^c}),h_R\rangle := \int_{(3I)^c}\int_I \langle [U_1(x_1,y_1)-U_1(c_I,y_1)]1_{3J},h_J\rangle h_I(x_1)\,\mathrm{d}x_1\,\mathrm{d}y_1,
$$

where

$$
\begin{aligned}
&\langle [U_1(x_1,y_1)-U_1(c_I,y_1)]1_{3J},h_J\rangle\\
&\quad := \int_{(3J)^c}\int_J (K_{U_1(x_1,y_1)-U_1(c_I,y_1)}(x_2,y_2)-K_{U_1(x_1,y_1)-U_1(c_I,y_1)}(c_J,y_2))h_J(x_2)\,\mathrm{d}x_2\,\mathrm{d}y_2.
\end{aligned}
$$

Now for this kernel we have

$$
\begin{aligned}
&|K_{U_1(x_1,y_1)-U_1(c_I,y_1)}(x_2,y_2)-K_{U_1(x_1,y_1)-U_1(c_I,y_1)}(c_J,y_2)|\\
&\qquad\lesssim \|U_1(x_1,y_1)-U_1(c_I,y_1)\|_{CZ_\alpha}\frac{\ell(J)^\alpha}{|c_J-y_2|^{m+\alpha}}\\
&\qquad\lesssim \frac{\ell(I)^\alpha}{|c_I-y_1|^{n+\alpha}}\frac{\ell(J)^\alpha}{|c_J-y_2|^{m+\alpha}}.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
&|\langle T(1_{(3I)^c}\otimes 1_{(3J)^c}),h_R\rangle|\\
&\qquad\lesssim \int_{(3I)^c}\int_{(3J)^c}|I|^{\frac12}|J|^{\frac12}\frac{\ell(I)^\alpha}{|c_I-y_1|^{n+\alpha}}\frac{\ell(J)^\alpha}{|c_J-y_2|^{m+\alpha}}\,\mathrm{d}y_2\,\mathrm{d}y_1\\
&\qquad\lesssim |I\times J|^{\frac12}<\infty.
\end{aligned}
$$

Before we get into the proof of the main result of this chapter, we show Journé's covering lemma.

4.3 Journé's lemma

First some notations which we are going to use in the proof of Journé's covering lemma. Let $\mathcal{D} = \mathcal{D}^n \times \mathcal{D}^m$, where $\mathcal{D}^n$ and $\mathcal{D}^m$ are some dyadic grids in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. For rectangles $R = I \times J \in \mathcal{D}$, we write $R^{(i,j)} := I^{(i)} \times J^{(j)}$ and

$$\mathrm{gen}_1(R) := \mathrm{gen}(I), \qquad \mathrm{gen}(R) := (\mathrm{gen}(I), \mathrm{gen}(J)),$$

where $I^{(i)}$ is the dyadic ancestor and $\mathrm{gen}(I)$ is the generation of a dyadic cube.

As before, we denote

$$\widetilde{\Omega} := \left\{ M_{\mathcal{D}}\, 1_\Omega > \frac{1}{2} \right\},$$

where $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ such that $|\Omega| < \infty$ and for all $x \in \Omega$ there exists $R \in \mathcal{D}$ so that $R \subset \Omega$. Thus, define the embeddedness of R in Ω as

$$\mathrm{emb}_1(R;\Omega) := \sup \left\{ k : R^{(k,0)} \subset \widetilde{\Omega} \right\}.$$

Furthermore, if for dyadic rectangle $R = I \times J \subset \Omega$ holds that $I \times \widetilde{J} \not\subset \Omega$ for any $\widetilde{J} \supsetneq J$, then we say that R is 2-maximal.

Theorem 4.3. *(Journé's lemma). Let $\omega : \mathbb{N} \to \mathbb{R}_+$ be a decreasing function with the property that $\sum_{k=0}^{\infty} \omega(k) < \infty$. Then there holds that*

$$\sum_{\substack{R \subset \Omega \\ 2\text{-maximal}}} \omega(\mathrm{emb}_1(R;\Omega)) \times |R| \le 2 \sum_{k=0}^{\infty} \omega(k) \times |\Omega|. \tag{4.4}$$

Proof. Let us define $\delta(k) := \omega(k) - \omega(k+1) \ge 0$. Thus we have that $\omega(j) = \sum_{k=j}^{\infty} \delta(k)$ and we can write the left hand side of the inequality (4.4) as

$$\begin{aligned}
\sum_{\substack{R \subset \Omega \\ 2\text{-maximal}}} \omega(\mathrm{emb}_1(R;\Omega)) \times |R| &= \sum_{j=0}^{\infty} \omega(j) \sum_{\substack{R \subset \Omega \\ 2\text{-maximal} \\ \mathrm{emb}_1(R;\Omega) = j}} |R| \\
&= \sum_{k=0}^{\infty} \delta(k) \sum_{\substack{R \subset \Omega \\ 2\text{-maximal} \\ \mathrm{emb}_1(R;\Omega) \le k}} |R| \\
&= \sum_{k=0}^{\infty} \delta(k) \sum_{i=0}^{k} \sum_{R \in \mathcal{R}(k,i)} |R|,
\end{aligned} \tag{4.5}$$

where $\mathcal{R}(k,i) := \{R \subset \Omega \quad 2\text{-maximal} : \mathrm{emb}_1(R;\Omega) \le k, \mathrm{gen}_1(R) \equiv i \mod k+1\}$.

Next, we want to define a subset for every $R = I \times J \in \mathcal{R}(k,i)$ such that the sets are pairwise disjoint. Noting that if $R = I \times J \in \mathcal{R}(k,i)$ and $R' = I' \times J' \in \mathcal{R}(k,i)$ intersects, then $I \cap I' \neq \emptyset$ and $J \cap J' \neq \emptyset$. Suppose $I = I'$ then by the 2-maximality $J = J'$ hence $R = R'$. Thus, we can assume $I \subsetneq I'$. Since $R = I \times J \not\subset I \times J' \subset R'$ by the 2-maximality of R, we have that $J \supset J'$.

Now this motivates us to define

$$E(R) := R \setminus \bigcup_{\substack{R'=I'\times J' \in \mathcal{R}(k,i)\\ I' \supsetneq I}} R' = I \times \left(J \setminus \bigcup_{\substack{I'\times J' \in \mathcal{R}(k,i)\\ I' \supsetneq I}} J' \right)$$

for every $R = I \times J \in \mathcal{R}(k,i)$ and we have that $J' \subset J$ by the arguments above.

Then we observe that for $R = I \times J \neq R' = I' \times J'$ in $\mathcal{R}(k,i)$ such that $R \cap R' \neq \emptyset$ either $I \subsetneq I'$ or $I \supsetneq I'$. Assuming $I \subsetneq I'$ we get that $E(R) \cap R' = \emptyset$. Thus $E(R)$ and $E(R')$ are pairwise disjoint.

Let $R = I \times J \in \mathcal{R}(k,i)$. Then we know that $\mathrm{emb}_1(R;\Omega) \le k$, that is, $R^{(k+1,0)} = I^{(k+1)} \times J \not\subset \widetilde{\Omega}$. Thus we have that

$$\begin{aligned} \frac{1}{2} &\ge \frac{1}{|R^{(k+1,0)}|} \int_{R^{(k+1,0)}} 1_\Omega \, \mathrm{d}x \\ &= \frac{|R^{(k+1,0)} \cap \Omega|}{|R^{(k+1,0)}|}, \end{aligned} \tag{4.6}$$

which implies that

$$\frac{1}{2}|R^{(k+1,0)}| \le |R^{(k+1,0)} \setminus \Omega|.$$

Clearly, if $I \subsetneq I'$ and $\mathrm{gen}(I) \equiv \mathrm{gen}(I') \mod k+1$, then $I^{(k+1)} \subset I'$. Thus

$$R^{(k+1,0)} \setminus \Omega \subset R^{(k+1,0)} \setminus \bigcup_{\substack{R'=I'\times J' \in \mathcal{R}(k,i)\\ I' \supsetneq I}} R' = I^{(k+1)} \times \left(J \setminus \bigcup_{\substack{I'\times J' \in \mathcal{R}(k,i)\\ I' \supsetneq I}} J' \right).$$

Hence, we obtain

$$\begin{aligned} \frac{1}{2}|I^{(k+1)}||J| &= \frac{1}{2}|R^{(k+1,0)}| \\ &\le |R^{(k+1,0)} \setminus \Omega| \\ &\le |I^{(k+1)}| \left| J \setminus \bigcup_{\substack{I'\times J' \in \mathcal{R}(k,i)\\ I' \supsetneq I}} J' \right|. \end{aligned} \tag{4.7}$$

Thus we get

$$|E(R)| = |I| \Big| J \setminus \bigcup_{\substack{I' \times J' \in \mathcal{R}(k,i) \\ I' \supsetneq I}} J' \Big| \geq \frac{1}{2}|R| = \frac{1}{2}|I||J|$$

for every $R \in \mathcal{R}(k,i)$.

Hence, we have estimate

$$\sum_{R \in \mathcal{R}(k,i)} |R| \leq \sum_{R \in \mathcal{R}(k,i)} 2|E(R)| \leq 2|\Omega|$$

by the fact that $E(R) \subset \Omega$ for all $R \in \mathcal{R}(k,i)$ and the sets $E(R)$ are pairwise disjoint. Using this estimate we conclude that

$$\sum_{\substack{R \subset \Omega \\ 2\text{-maximal}}} \omega(\mathrm{emb}_1(R;\Omega)) \times |R| \leq 2 \sum_{k=0}^{\infty} \delta(k) \times (k+1) \times |\Omega| \leq 2 \sum_{k=0}^{\infty} \omega(k) \times |\Omega|$$

as desired. □

4.4 Boundedness of $T1$

Next theorem will show that the sequence $(\langle T1, h_R \rangle)_{R \in \mathcal{D}} \in \mathrm{BMO}_{prod}(\mathcal{D})$ for every $\mathcal{D}$, that is, $T1 \in \mathrm{BMO}_{prod}(\mathbb{R}^{n+m})$.

Theorem 4.8. *Suppose that T is a bi-parameter singular integral satisfying assumptions above. Then*

$$\sum_{\substack{R = I \times J \in \mathcal{D} \\ R \subset \Omega}} |\langle T1, h_R \rangle|^2 \lesssim |\Omega|, \tag{4.9}$$

where $\mathcal{D} = \mathcal{D}^n \times \mathcal{D}^m$ and $h_R = h_I \otimes h_J$, for every dyadic grid $\mathcal{D}^n, \mathcal{D}^m$ and all open sets $\Omega \subset \mathbb{R}^{n+m}$ with $|\Omega| < \infty$. Here the constant depends on $T : L^2 \to L^2$.

Proof. Let $\mathcal{D} = \mathcal{D}^n \times \mathcal{D}^m$ be an arbitrary dyadic grid on $\mathbb{R}^{n+m}$. Let $\Omega \subset \mathbb{R}^{n+m}$ be a bounded set such that for every $x \in \Omega$ there exists $R \in \mathcal{D}$ so that $x \in R \subset \Omega$. Let us denote $\widetilde{\Omega} := \{M_{\mathcal{D}} 1_\Omega > 1/2\}$ and $\widehat{\Omega} := \{M 1_{\widetilde{\Omega}} > 2^{-n-m-1}\}$. Recall that by Chebyshev's inequality and Lemma 2.4 we have that

$$|\widehat{\Omega}| \lesssim \|M 1_{\widetilde{\Omega}}\|_{L^2}^2 \lesssim \|1_{\widetilde{\Omega}}\|_{L^2}^2 \leq |\widetilde{\Omega}| \lesssim \|M_{\mathcal{D}} 1_\Omega\|_{L^2}^2 \lesssim \|1_\Omega\|_{L^2}^2 \leq |\Omega|.$$

First, we write

$$\sum_{\substack{R=I\times J\in\mathcal{D}\\R\subset\Omega}} |\langle T1, h_R\rangle|^2 = \sum_{\substack{R=I\times J\in\mathcal{D}\\R\subset\Omega}} |\langle T1_{\widehat{\Omega}}, h_R\rangle + \langle T1_{\widehat{\Omega}^c}, h_R\rangle|^2$$
$$\lesssim \sum_{\substack{R=I\times J\in\mathcal{D}\\R\subset\Omega}} |\langle T1_{\widehat{\Omega}}, h_R\rangle|^2 + \sum_{\substack{R=I\times J\in\mathcal{D}\\R\subset\Omega}} |\langle T1_{\widehat{\Omega}^c}, h_R\rangle|^2.$$

We observe that

$$\sum_{\substack{R=I\times J\in\mathcal{D}\\R\subset\Omega}} |\langle T1_{\widehat{\Omega}}, h_R\rangle|^2 \le \left\|T1_{\widehat{\Omega}}\right\|^2_{L^2(\mathbb{R}^{n+m})} \lesssim \left\|1_{\widehat{\Omega}}\right\|^2_{L^2(\mathbb{R}^{n+m})} \lesssim |\widehat{\Omega}| \lesssim |\Omega|.$$

Hence, it is enough to show that

$$\sum_{\substack{R=I\times J\in\mathcal{D}\\R\subset\Omega}} |\langle T1_{\widehat{\Omega}^c}, h_R\rangle|^2 \lesssim |\Omega|.$$

For every $J\in\mathcal{D}^m$ let $\mathcal{F}_J$ be a collection of maximal $F\in\mathcal{D}^n$ such that $F\times J\subset\widetilde{\Omega}$ and denote $F_J := \bigcup_{F\in\mathcal{F}_J} 2F$. Now we write

$$\sum_{\substack{R=I\times J\in\mathcal{D}\\R\subset\Omega}} |\langle T1_{\widehat{\Omega}^c}, h_R\rangle|^2 \lesssim \sum_{\substack{R=I\times J\in\mathcal{D}\\R\subset\Omega}} \Big(|\langle T1_{\widehat{\Omega}^c}1_{F_J}, h_R\rangle|^2 + |\langle T1_{\widehat{\Omega}^c}1_{F_J^c}, h_R\rangle|^2\Big).$$

First, we estimate the first term

$$\sum_{\substack{R=I\times J\in\mathcal{D}\\R\subset\Omega}} |\langle T1_{\widehat{\Omega}^c}1_{F_J}, h_R\rangle|^2 \le \sum_{J\in\mathcal{D}^m}\sum_{I\in\mathcal{D}^n} |\langle T1_{\widehat{\Omega}^c}1_{F_J}, h_R\rangle|^2.$$

Observe that if $I\times J\subset\widetilde{\Omega}$, then

$$M1_{\widetilde{\Omega}}(x_1,x_2) \ge \frac{|(2I\times 2J)\cap\widetilde{\Omega}|}{|2I\times 2J|} \ge \frac{|(I\times J)\cap\widetilde{\Omega}|}{|2I\times 2J|} = 2^{-n-m}\frac{|I\times J|}{|I\times J|} > 2^{-n-m-1}$$

for every $(x_1,x_2)\in 2I\times 2J$, that is, $2I\times 2J\subset\widehat{\Omega}$. Thus $\bigcup_{J'}(F_{J'}\times 2J')\subset\widehat{\Omega}$, implying that $\operatorname{spt} 1_{\widehat{\Omega}^c}1_{F_J}\subset F_J\times(2J)^c$ and therefore $\operatorname{spt}_{\mathbb{R}^m} 1_{\widehat{\Omega}^c}1_{F_J}\cap\operatorname{spt}_{\mathbb{R}^m} h_R=\emptyset$. By the assumptions of T we write

$$\langle T1_{\widehat{\Omega}^c}1_{F_J}, h_R\rangle = \iint_{\mathbb{R}^{2m}} \langle B_J(x_2,y_2), h_I\rangle h_J(x_2)\,\mathrm{d}x_2\,\mathrm{d}y_2,$$

where $B_J(x_2, y_2) := [U_2(x_2, y_2) - U_2(c_J, y_2)](1_{\widehat{\Omega}^c}(\cdot, y_2)1_{F_J})$. Now by Minkowski's integral inequality

$$\begin{aligned}\sum_{I\in\mathcal{D}^n} |\langle T1_{\widehat{\Omega}^c}1_{F_J}, h_R\rangle|^2 &= \sum_{I\in\mathcal{D}^n} \left| \iint_{\mathbb{R}^{2m}} \langle B_J(x_2, y_2), h_I\rangle h_J(x_2)\,\mathrm{d}x_2\,\mathrm{d}y_2 \right|^2 \\ &\leq \left(\iint_{\mathbb{R}^{2m}} \left(\sum_{I\in\mathcal{D}^n} \left|\langle B_J(x_2, y_2), h_I\rangle h_J(x_2)\right|^2 \right)^{\frac{1}{2}} \mathrm{d}x_2\,\mathrm{d}y_2 \right)^2 \\ &\leq \left(\iint_{\mathbb{R}^{2m}} \frac{1_J(x_2)}{|J|^{\frac{1}{2}}} \left(\sum_{I\in\mathcal{D}^n} \left|\langle B_J(x_2, y_2), h_I\rangle\right|^2 \right)^{\frac{1}{2}} \mathrm{d}x_2\,\mathrm{d}y_2 \right)^2.\end{aligned}$$

Then using the $L^2 \to L^2$ boundedness of $U_2(x_2, y_2) - U_2(c_J, y_2)$ we get

$$\begin{aligned}\left(\sum_{I\in\mathcal{D}^n} |\langle B_J(x_2, y_2), h_I\rangle|^2 \right)^{\frac{1}{2}} &\leq \|B_J(x_2, y_2)\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \left\|U_2(x_2, y_2) - U_2(c_J, y_2)\right\| \left\|(1_{\widehat{\Omega}^c}(\cdot, y_2)1_{F_J})\right\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \frac{\ell(J)^\alpha}{|c_J - y_2|^{m+\alpha}} \left\|(1_{\widehat{\Omega}^c}(\cdot, y_2)1_{F_J})\right\|_{L^2(\mathbb{R}^n)}\end{aligned}$$

for $x_2 \in J$ and $y_2 \in (2J)^c$.

Hence, we obtain

$$\sum_{J\in\mathcal{D}^m}\sum_{I\in\mathcal{D}^n} |\langle T1_{\widehat{\Omega}}1_{F_J}, h_R\rangle|^2 \lesssim \sum_{J\in\mathcal{D}^m} \left(|J|^{\frac{1}{2}} \int_{\mathbb{R}^m} \frac{\ell(J)^\alpha}{|c_J - y_2|^{m+\alpha}} \left\|(1_{\widehat{\Omega}^c}(\cdot, y_2)1_{F_J})\right\|_{L^2(\mathbb{R}^n)} \mathrm{d}y_2 \right)^2.$$

Then applying Hölder's inequality for the integral

$$\begin{aligned}&\left(\int_{(2J)^c} \frac{\ell(J)^{\frac{\alpha}{2}}}{|c_J - y_2|^{\frac{m+\alpha}{2}}} \frac{\ell(J)^{\frac{\alpha}{2}}}{|c_J - y_2|^{\frac{m+\alpha}{2}}} \left\|(1_{\widehat{\Omega}^c}(\cdot, y_2)1_{F_J})\right\|_{L^2(\mathbb{R}^n)} \mathrm{d}y_2 \right)^2 \\ &\quad\leq \left(\int_{(2J)^c} \frac{\ell(J)^\alpha}{|c_J - y_2|^{m+\alpha}}\,\mathrm{d}y_2 \right) \left(\int_{(2J)^c} \frac{\ell(J)^\alpha}{|c_J - y_2|^{m+\alpha}} \left\|(1_{\widehat{\Omega}^c}(\cdot, y_2)1_{F_J})\right\|_{L^2(\mathbb{R}^n)}^2 \mathrm{d}y_2 \right),\end{aligned}$$

where the first integral can be computed summing integrals over the annuli
$A(j) := B(c_J, \ell(J)2^{j+1}) \setminus B(c_J, \ell(J)2^j)$ as we computed in the one-parameter case.

Now by Fubini's theorem, we write

$$
\begin{aligned}
&\sum_{J\in\mathcal{D}^m}\sum_{I\in\mathcal{D}^n}|\langle T1_{\widehat{\Omega}^c}1_{F_J},h_R\rangle|^2\\
&\qquad\lesssim\sum_{J\in\mathcal{D}^m}|J|\int_{\mathbb{R}^m}\frac{\ell(J)^\alpha}{|c_J-y_2|^{m+\alpha}}\left\|(1_{\widehat{\Omega}^c}(\cdot,y_2)1_{F_J})\right\|^2_{L^2(\mathbb{R}^n)}\,\mathrm{d}y_2\\
&\qquad=\sum_{J\in\mathcal{D}^m}\iint 1_{F_J\times J}(y_1,x_2)\int\frac{\ell(J)^\alpha}{|c_J-y_2|^{m+\alpha}}1_{\widehat{\Omega}^c}(y_1,y_2)\,\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x_2\\
&\qquad=\iint_{\bigcup_{J'}F'_J\times J'}\Bigg(\sum_{\substack{J\in\mathcal{D}^m\\(y_1,x_2)\in F_J\times J}}\int\frac{\ell(J)^\alpha}{|c_J-y_2|^{m+\alpha}}1_{\widehat{\Omega}^c}(y_1,y_2)\,\mathrm{d}y_2\Bigg)\,\mathrm{d}y_1\,\mathrm{d}x_2.
\end{aligned}
\tag{4.10}
$$

Let $V=V(y_1,x_2)$ be the maximal $J\in\mathcal{D}^m$ for which $(y_1,x_2)\in F_J\times J$. Since we have $|c_V-c_J|\le\ell(V)/2\le|c_J-y_2|$ for $J\subset V$ and $y_2\in(2V)^c$, we get $|c_V-y_2|\le|c_V-c_J|+|c_J-y_2|\le\ell(V)/2+|c_J-y_2|\le2|c_J-y_2|$. Thus we can estimate

$$
\int\frac{\ell(J)^\alpha}{|c_J-y_2|^{m+\alpha}}1_{\widehat{\Omega}^c}(y_1,y_2)\,\mathrm{d}y_2\lesssim\int_{V^c}\frac{\ell(J)^\alpha}{|c_V-y_2|^{m+\alpha}}\,\mathrm{d}y_2\lesssim\left(\frac{\ell(J)}{\ell(V)}\right)^\alpha.
$$

Since we know that there exist a unique dyadic cube $J_k(x_2)\subset V$ for all $k\le\ell(V)$, we have

$$
\sum_{\substack{J\in\mathcal{D}^m\\(y_1,x_2)\in F_J\times J}}\int\frac{\ell(J)^\alpha}{|c_J-y_2|^{m+\alpha}}1_{\widehat{\Omega}^c}(y_1,y_2)\,\mathrm{d}y_2\lesssim\sum_{\substack{J\in\mathcal{D}^m\\x_2\in J\subset V}}\left(\frac{\ell(J)}{\ell(V)}\right)^\alpha\le\sum_{k=0}^\infty\left(\frac{1}{2}\right)^{\alpha k}\lesssim1.
$$

Continuing from (4.10) we have

$$
\sum_{J\in\mathcal{D}^m}\sum_{I\in\mathcal{D}^n}|\langle T1_{\widehat{\Omega}^c}1_{F_J},h_R\rangle|^2\lesssim|\bigcup_{J'}F_{J'}\times J'|\lesssim|\widehat{\Omega}|\lesssim|\Omega|
$$

by the fact that $\bigcup_{J'}F_{J'}\times J'\subset\widehat{\Omega}$.

Lastly, it remains to estimate

$$
\sum_{\substack{R=I\times J\in\mathcal{D}\\R\subset\Omega}}|\langle T1_{\widehat{\Omega}^c}1_{F_J^c},h_R\rangle|^2=\sum_{I\in\mathcal{D}^n}\sum_{\substack{J\in\mathcal{D}^m\\I\times J\subset\Omega}}|\langle T1_{\widehat{\Omega}^c}1_{F_J^c},h_R\rangle|^2,
$$

now summing in different order than in the previous term.

Let $\mathcal{G}_I$ be a collection of maximal $G\in\mathcal{D}^m$ such that $I\times G\subset\Omega$. Given such a G let I_G be the maximal parent of I for which $I_G\times G\subset\widetilde{\Omega}$. Then we write

$$
\sum_{\substack{J\in\mathcal{D}^m\\I\times J\subset\Omega}}|\langle T1_{\widehat{\Omega}^c}1_{F_J^c},h_R\rangle|^2=\sum_{G\in\mathcal{G}_I}\sum_{\substack{J\in\mathcal{D}^m\\J\subset G}}|\langle T1_{\widehat{\Omega}^c}1_{F_J^c},h_R\rangle|^2.
$$

Since $I_G \times J \subset I_G \times G \subset \widetilde{\Omega}$ there exists $F \in \mathcal{F}_J$ such that $I_G \subset F$ for which $2I_G \subset 2F \subset F_J$, we have that $F_J^c \subset (2I_G)^c \subset (2I)^c$. Thus we may write

$$\langle T1_{\widehat{\Omega}^c} 1_{F_J^c}, h_R \rangle = \iint_{\mathbb{R}^{2n}} \langle B_I(x_1, y_1), h_J \rangle 1_{F_J^c}(y_1) h_I(x_1) \, \mathrm{d}x_1 \, \mathrm{d}y_1,$$

where $B_I(x_1, y_1) := \big[U_1(x_1, y_1) - U_1(c_I, y_1)\big](1_{\widehat{\Omega}^c}(y_1, \cdot))$.

By Minkowski's integral inequality we get

$$\begin{aligned}
&\sum_{\substack{J \in \mathcal{D}^m \\ J \subset G}} |\langle T1_{\widehat{\Omega}^c} 1_{F_J^c}, h_R \rangle|^2 \\
&\quad \le \bigg(\iint_{\mathbb{R}^{2n}} \frac{1_I(x_1)}{|I|^{\frac{1}{2}}} \bigg(\sum_{\substack{J \in \mathcal{D}^m \\ J \subset G}} |\langle B_I(x_1, y_1), h_J \rangle|^2 |1_{F_J^c}(y_1)|^2 \bigg)^{\frac{1}{2}} \mathrm{d}x_1 \, \mathrm{d}y_1 \bigg)^2 \\
&\quad \le \bigg(\iint_{\mathbb{R}^{2n}} \frac{1_I(x_1)}{|I|^{\frac{1}{2}}} 1_{(2I_G)^c}(y_1) \bigg(\sum_{\substack{J \in \mathcal{D}^m \\ J \subset G}} |\langle B_I(x_1, y_1), h_J \rangle|^2 \bigg)^{\frac{1}{2}} \mathrm{d}x_1 \, \mathrm{d}y_1 \bigg)^2 .
\end{aligned} \tag{4.11}$$

By Theorem 4.1 we have

$$\begin{aligned}
\bigg(\sum_{\substack{J \in \mathcal{D}^m \\ J \subset G}} |\langle B_I(x_1, y_1), h_J \rangle|^2 \bigg)^{\frac{1}{2}} &\le \bigg(\sum_{J \in \mathcal{D}^m} |\langle 1_G \big(B_I(x_1, y_1) - \langle B_I(x_1, y_1) \rangle_G \big), h_J \rangle|^2 \bigg)^{\frac{1}{2}} \\
&= \big\| 1_G \big(B_I(x_1, y_1) - \langle B_I(x_1, y_1) \rangle_G \big) \big\|_{L^2} \\
&\lesssim |G|^{\frac{1}{2}} \, \| B_I(x_1, y_1) \|_{\mathrm{BMO}_2(\mathcal{D}^m)} \\
&\lesssim |G|^{\frac{1}{2}} \, \big\| 1_{\widehat{\Omega}} \big\|_{L^\infty} \, \| U_1(x_1, y_1) - U_1(c_I, y_1) \| \\
&\lesssim |G|^{\frac{1}{2}} \frac{\ell(I)^\alpha}{|c_I - y_1|^{n+\alpha}}
\end{aligned}$$

for $x_1 \in I$ and $y_1 \in (2I_G)^c \subset (2I)^c$.

As previously with the other term Hölder's inequality gives us

$$
\begin{aligned}
&\sum_{I\in\mathcal{D}^n}\sum_{G\in\mathcal{G}_I}\sum_{\substack{J\in\mathcal{D}^m\\ J\subset G}}|\langle T1_{\widehat{\Omega}^c}1_{F_J^c},h_R\rangle|^2\\
&\qquad\le \sum_{I\in\mathcal{D}^n}|I|\sum_{G\in\mathcal{G}_I}\left(\int_{\mathbb{R}^n}1_{(2I_G)^c}(y_1)|G|^{\frac12}\frac{\ell(I)^\alpha}{|c_I-y_1|^{n+\alpha}}\,\mathrm{d}y_1\right)^2\\
&\qquad\lesssim \sum_{I\in\mathcal{D}^n}|I|\sum_{G\in\mathcal{G}_I}\int_{\mathbb{R}^n}1_{(2I_G)^c}(y_1)|G|\frac{\ell(I)^\alpha}{|c_I-y_1|^{n+\alpha}}\,\mathrm{d}y_1\\
&\qquad= \sum_{I\in\mathcal{D}^n}|I|\sum_{G\in\mathcal{G}_I}|G|\int_{\mathbb{R}^n}1_{(2I_G)^c}(y_1)\frac{\ell(I)^\alpha}{|c_I-y_1|^{n+\alpha}}\,\mathrm{d}y_1
\end{aligned}
$$

Since $y_1\in(2I_G)^c$, we have that $|c_I-y_1|\ge 1/2|c_{I_G}-y_1|$. Now computing the integral similarly as before.

$$
\int_{\mathbb{R}^n}1_{(2I_G)^c}(y_1)\frac{\ell(I)^\alpha}{|c_{I_G}-y_1|^{n+\alpha}}\,\mathrm{d}y_1\lesssim\left(\frac{\ell(I)}{\ell(I_G)}\right)^\alpha. \tag{4.12}
$$

Thus we have

$$
\begin{aligned}
\sum_{I\in\mathcal{D}^n}\sum_{G\in\mathcal{G}_I}\sum_{\substack{J\in\mathcal{D}^m\\ J\subset G}}|\langle T1_{\widehat{\Omega}^c}1_{F_J^c},h_R\rangle|^2 &\lesssim \sum_{I\in\mathcal{D}^n}\sum_{G\in\mathcal{G}_I}|I\times G|\left(\frac{\ell(I)}{\ell(I_G)}\right)^\alpha\\
&\le \sum_{\substack{I\times G\subset\Omega\\ 2\text{-maximal}}}2^{-\alpha\,\mathrm{emb}_1(I\times G;\Omega)}|I\times G|\\
&\lesssim|\Omega|,
\end{aligned} \tag{4.13}
$$

where last step is by Theorem 4.3. $\square$

Chapter 5

Boundedness of bi-parameter paraproducts

Now we study bi-parameter paraproducts which are essential for proving the $L^2 \to L^2$ boundedness of the bi-parameter Calderón-Zygmund operators. Since these $T1$ theorems are out of the scope of this thesis, we only study the boundedness property of these paraproducts.

Let $(\lambda_R)_R$ be a sequence indexed over dyadic rectangles in $\mathcal{D} = \mathcal{D}^n \times \mathcal{D}^n$. We say that Π_λ is a paraproduct if it has one of the following four forms:

(1)
$$\Pi_\lambda f = \sum_{R \in \mathcal{D}} \lambda_R \Big\langle f, \frac{1_R}{|R|} \Big\rangle h_R = \sum_{R \in \mathcal{D}} \lambda_R \Big\langle f, \frac{1_I \otimes 1_J}{|I \times J|} \Big\rangle h_I \otimes h_J$$

(2)
$$\Pi_\lambda f = \sum_{R \in \mathcal{D}} \lambda_R \langle f, h_R \rangle \frac{1_R}{|R|}$$

(3)
$$\Pi_\lambda f = \sum_{R \in \mathcal{D}} \lambda_R \Big\langle f, h_I \otimes \frac{1_J}{|J|} \Big\rangle \frac{1_I}{|I|} \otimes h_J$$

(4)
$$\Pi_\lambda f = \sum_{R \in \mathcal{D}} \lambda_R \Big\langle f, \frac{1_I}{|I|} \otimes h_J \Big\rangle h_I \otimes \frac{1_J}{|J|}$$

where $f : \mathbb{R}^{n+m} \to \mathbb{C}$.

First, we consider paraproducts of the form (1) or (2). For these we show that it is necessary that the sequence λ is in the product BMO space.

Theorem 5.1. *If Π_λ has the form* (1) *or* (2) *then for all $f \in L^2(\mathbb{R}^{n+m})$*

$$\|\Pi_\lambda f\|_{L^2(\mathbb{R}^{n+m})} \lesssim \|f\|_{L^2(\mathbb{R}^{n+m})}$$

if and only if $\|\lambda\|_{\mathrm{BMO}_{prod}(\mathcal{D})} < \infty$.

Proof. Let us first assume that $\|\lambda\|_{\mathrm{BMO}_{prod}} < \infty$. It is enough to show that

$$|\langle \Pi_\lambda f, g\rangle| \lesssim \|\lambda\|_{\mathrm{BMO}_{prod}} \|f\|_{L^2(\mathbb{R}^{n+m})} \|g\|_{L^2(\mathbb{R}^{n+m})}$$

when Π_λ has the form (1) since

$$\langle \Pi^1_\lambda f, g\rangle = \sum_{R\in\mathcal{D}} \lambda_R \langle f\rangle_R \langle g, h_R\rangle = \langle f, \Pi^2_\lambda g\rangle.$$

By Lemma 3.17, we have

$$\begin{aligned}
|\langle \Pi_\lambda f, g\rangle| &\leq \int \Big| \sum_{R\in\mathcal{D}} \lambda_R \Big\langle f, \frac{1_R}{|R|}\Big\rangle h_R\, g\Big| \\
&\leq \sum_{R\in\mathcal{D}} |\lambda_R| \langle |f|\rangle_R |\langle g, h_R\rangle| \\
&\lesssim \|\lambda\|_{\mathrm{BMO}_{prod}} \int \left(\sum_{R\in\mathcal{D}} \langle |f|\rangle_R^2 |\langle g, h_R\rangle|^2 \frac{1_R}{|R|}\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus we can estimate

$$\begin{aligned}
\int_{\mathbb{R}^{n+m}} \left(\sum_{R\in\mathcal{D}} \langle |f|\rangle_R^2 |\langle g, h_R\rangle|^2 \frac{1_R}{|R|}\right)^{\frac{1}{2}} &\leq \int_{\mathbb{R}^{n+m}} M|f| \left(\sum_{R\in\mathcal{D}} |\langle g, h_R\rangle|^2 \frac{1_R}{|R|}\right)^{\frac{1}{2}} \\
&\leq \|Mf\|_{L^2(\mathbb{R}^{n+m})} \left(\int_{\mathbb{R}^{n+m}} \sum_{R\in\mathcal{D}} |\langle g, h_R\rangle|^2 \frac{1_R}{|R|}\right)^{\frac{1}{2}} \\
&\lesssim \|f\|_{L^2(\mathbb{R}^{n+m})} \left(\sum_{R\in\mathcal{D}} |\langle g, h_R\rangle|^2\right)^{\frac{1}{2}} \\
&\leq \|f\|_{L^2(\mathbb{R}^{n+m})} \|g\|_{L^2(\mathbb{R}^{n+m})}.
\end{aligned}$$

Then the converse claim. Assume that

$$\|\Pi_\lambda f\|_{L^2(\mathbb{R}^{n+m})} \lesssim \|f\|_{L^2(\mathbb{R}^{n+m})}$$

for all $f \in L^2(\mathbb{R}^{n+m})$. Observe that

$$\begin{aligned}\|\Pi_\lambda f\|^2_{L^2(\mathbb{R}^{n+m})} &= \sum_{R\in\mathcal{D}} |\langle \Pi_\lambda f, h_R\rangle|^2 \\ &= \sum_{R\in\mathcal{D}} |\langle \sum_{R'\in\mathcal{D}} \lambda_{R'}\langle f\rangle_{R'} h_{R'}, h_R\rangle|^2 \\ &= \sum_{R\in\mathcal{D}} |\lambda_R \langle f\rangle_R|^2.\end{aligned}$$

Thus we get that

$$\sum_{\substack{R\in\mathcal{D}\\ R\subset\Omega}} |\lambda_R|^2 |\langle f\rangle_R|^2 \leq \sum_{R\in\mathcal{D}} |\lambda_R|^2 |\langle f\rangle_R|^2 \lesssim \|f\|^2_{L^2}$$

for fixed $\Omega \subsetneq \mathbb{R}^{n+m}$, then choosing $f = 1_\Omega$ we get

$$\sum_{\substack{R\in\mathcal{D}\\ R\subset\Omega}} |\lambda_R|^2 \lesssim |\Omega|.$$

Hence, we conclude that

$$\|\lambda\|_{\mathrm{BMO}_{prod}(\mathcal{D})} \lesssim 1.$$

□

Next, we study the paraproducts of the form (3) or (4). We show that if the sequence λ belongs to the product BMO space, then the paraproduct is bounded. Althought, the boundedness does not imply that the sequence is necessarily in the product BMO space or even in rectangular one as shown by Martikainen and Orponen [5].

Theorem 5.2. *If* Π_λ *has the form* (3) *or* (4) *and* $\|\lambda\|_{\mathrm{BMO}_{prod}(\mathcal{D})} < \infty$, *then*

$$\|\Pi_\lambda f\|_{L^2(\mathbb{R}^{n+m})} \lesssim \|\lambda\|_{\mathrm{BMO}_{prod}(\mathcal{D})} \|f\|_{L^2(\mathbb{R}^{n+m})}.$$

Proof. Let Π_λ have the form (3). Similarly as before we get

$$\begin{aligned}|\langle \Pi_\lambda f, g\rangle| &\leq \sum_{R\in\mathcal{D}} |\lambda_R| \Big|\Big\langle f, h_I \otimes \frac{1_J}{|J|}\Big\rangle\Big| \Big|\Big\langle g, \frac{1_I}{|I|} \otimes h_J\Big\rangle\Big| \\ &\lesssim \|\lambda\|_{\mathrm{BMO}_{prod}} \int_{\mathbb{R}^{n+m}} \left(\sum_{R\in\mathcal{D}} \Big|\Big\langle f, h_I \otimes \frac{1_J}{|J|}\Big\rangle\Big|^2 \Big|\Big\langle g, \frac{1_I}{|I|} \otimes h_J\Big\rangle\Big|^2 \frac{1_I \otimes 1_J}{|I\times J|}\right)^{\frac{1}{2}}.\end{aligned}$$

Thus we get

$$
\begin{aligned}
&\int_{\mathbb{R}^{n+m}} \Bigg(\sum_{R\in\mathcal{D}} \Big|\Big\langle f, h_I \otimes \frac{1_J}{|J|}\Big\rangle\Big|^2 \Big|\Big\langle g, \frac{1_I}{|I|} \otimes h_J\Big\rangle\Big|^2 \frac{1_I \otimes 1_J}{|I\times J|}\Bigg)^{\frac{1}{2}} \\
&\qquad\qquad \le \int_{\mathbb{R}^{n+m}} \Bigg(\sum_{R\in\mathcal{D}} [M_{\mathcal{D}^m}\langle f, h_I\rangle_1]^2 \otimes [M_{\mathcal{D}^n}\langle g, h_J\rangle_2]^2 \frac{1_I \otimes 1_J}{|I\times J|}\Bigg)^{\frac{1}{2}} \\
&\qquad\qquad = \int_{\mathbb{R}^{n+m}} \Bigg(\sum_{I\in\mathcal{D}^n} [M_{\mathcal{D}^m}\langle f, h_I\rangle_1]^2 \otimes \frac{1_I}{|I|}\Bigg)^{\frac{1}{2}} \Bigg(\sum_{J\in\mathcal{D}^m} [M_{\mathcal{D}^n}\langle g, h_J\rangle_2]^2 \otimes \frac{1_J}{|J|}\Bigg)^{\frac{1}{2}} \\
&\qquad\qquad \le \Bigg(\int_{\mathbb{R}^{n+m}} \sum_{I\in\mathcal{D}^n} [M_{\mathcal{D}^m}\langle f, h_I\rangle_1]^2 \otimes \frac{1_I}{|I|}\Bigg)^{\frac{1}{2}} \Bigg(\int_{\mathbb{R}^{n+m}} \sum_{J\in\mathcal{D}^m} [M_{\mathcal{D}^n}\langle g, h_J\rangle_2]^2 \otimes \frac{1_J}{|J|}\Bigg)^{\frac{1}{2}} \\
&\qquad\qquad = \Bigg(\sum_{I\in\mathcal{D}^n} \|M_{\mathcal{D}^m}\langle f, h_I\rangle_1\|_{L^2(\mathbb{R}^m)}^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{J\in\mathcal{D}^m} \|M_{\mathcal{D}^n}\langle g, h_J\rangle_2\|_{L^2(\mathbb{R}^n)}^2\Bigg)^{\frac{1}{2}} \\
&\qquad\qquad \lesssim \Bigg(\sum_{I\in\mathcal{D}^n} \|\langle f, h_I\rangle_1\|_{L^2(\mathbb{R}^m)}^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{J\in\mathcal{D}^m} \|\langle g, h_J\rangle_2\|_{L^2(\mathbb{R}^n)}^2\Bigg)^{\frac{1}{2}}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\|f\|_{L^2(\mathbb{R}^{n+m})}^2 &= \int_{\mathbb{R}^m} \|f(\cdot, x_2)\|_{L^2(\mathbb{R}^n)}^2 \,\mathrm{d}x_2 \\
&= \int_{\mathbb{R}^m} \sum_{I\in\mathcal{D}^n} |\langle f(\cdot, x_2), h_I\rangle|^2 \,\mathrm{d}x_2 \\
&= \sum_{I\in\mathcal{D}^n} \int_{\mathbb{R}^m} |\langle f(\cdot, x_2), h_I\rangle|^2 \,\mathrm{d}x_2 \\
&= \sum_{I\in\mathcal{D}^n} \|\langle f, h_I\rangle_1\|_{L^2(\mathbb{R}^m)}^2 .
\end{aligned}
$$

Hence, we conclude

$$
\Bigg(\sum_{I\in\mathcal{D}^n} \|\langle f, h_I\rangle_1\|_{L^2(\mathbb{R}^m)}^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{J\in\mathcal{D}^m} \|\langle g, h_J\rangle_2\|_{L^2(\mathbb{R}^n)}^2\Bigg)^{\frac{1}{2}} = \|f\|_{L^2(\mathbb{R}^{n+m})} \|g\|_{L^2(\mathbb{R}^{n+m})} .
$$

□

Bibliography

[1] T. Hytönen and H. Martikainen. Non-homogeneous $T1$ theorem for bi-parameter singular integrals. *Advances in Mathematics*, 261:220–273, Feb. 2014.

[2] J.-L. Journé. Calderón-zygmund operators on product spaces. *Revista Mathemática Iberoammericana*, 1:55–91, 1985.

[3] J.-L. Journé. A covering lemma for product spaces. *Proceedings of the American Mathematical Society*, 96:593–598, 1986.

[4] K. Li, H. Martikainen, and E. Vuorinen. Bilinear bi-parameter singular integrals: Representation theorem and boundedness properties. *preprint*, Dec. 2017.

[5] H. Martikainen and T. Orponen. Some obstacles in characterising the boundedness of bi-parameter singular integrals. *Mathematische Zeitschrift*, 282:535 – 545, Apr. 2016.

[6] C. Muscalu and W. Schlag. *Classical and Multilinear Harmonic Analysis, Vol. II*. Cambridge Studies in Advanced Mathematics 138. Cambridge University Press, 2013.